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THE GEOMETRY OF THE ZEROS

OF

A POLYNOMIAL IN A COMPLEX VARIABLE

BY

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PREFACE

The subject treated in this book is sometimes called the *Analytic Theory of Polynomials* or the *Analytic Theory of Equations*. The word analytic is intended to suggest a study of equations from a non-algebraic standpoint. Since, however, the point of view is largely that of the geometric theory of functions of a complex variable, we have preferred to use the title of the *Geometry of the Zeros of a Polynomial in a Complex Variable*.

The connection of our subject with the geometric theory of functions of a complex variable becomes clear when we examine the type of problems treated in the subject and the type of methods used in solving these problems.

The problems center very largely about the study of the zeros of a polynomial $f(z)$ as functions of various parameters. The parameters are usually the coefficients of $f(z)$, or the zeros or the coefficients of some related polynomial $g(z)$. Regarded as points in the complex plane, the parameters are allowed to vary within certain prescribed regions. The corresponding locus R of the zeros of $f(z)$ is then to be determined. The locus R may consist of several non-overlapping regions $R_1, R_2, \cdots, R_\nu$. If so, we might ask how many zeros are contained in each R_k or in a specified subset of the R_k or, conversely, what subset of the R contains a prescribed number of zeros of $f(z)$. It may happen that the determination of the exact locus R may be too difficult, too complicated, or for some reason unnecessary. If so, we may wish to replace R by a simpler region S containing R. If for example S is chosen as a circle with center at the origin, its radius would of course furnish an upper bound to the moduli of the zeros of $f(z)$.

We may consider these questions regarding the locus R as pertaining to the geometric theory of functions for at least two reasons. First, we recognize that they are essentially questions concerning the mapping properties of the zeros viewed as analytic functions of the given parameters. Secondly, we recognize that, in determining the zeros of a polynomial $f(z)$, we are finding the A-points of the polynomial $g(z) = f(z) + A$; that is, the points where the polynomial $g(z)$ assumes a given value A. In other words, we may regard our problems as instances of the general problem of the value distribution of analytic functions. In fact, the solution to our problem may contribute to the solution of the general problem. For, if $G(z)$ is an arbitrary analytic function, we may be able to construct a sequence of polynomials $F_n(z)$ which in some region R converge uniformly to the function $F(z) = G(z) - A$; the zeros of $F(z)$, that is, the A-points of $G(z)$, may be then sought in R as the limit points of the zeros of the $F_n(z)$.

Our methods for investigating these questions will involve mostly the geometric operations with complex numbers and certain principles which are based

upon these operations and which are stated in Sec. 1. Among these is the principle that a sum of vectors cannot vanish if the vectors are all drawn from a point O on a line L to points all on the same side of L. Among these also is the so-called Principle of Argument and its corollaries such as the Rouché Theorem, the Cauchy Index Theorem, the theorem on the continuity of the zeros and the Hurwitz Theorem. Thus, due to the nature of not only its problems but also its methods, our subject may be considered as belonging to the geometric theory of functions.

Historically speaking, our subject dates from about the time when the geometric representation of complex numbers was introduced into mathematics. The first contributors to the subject were Gauss and Cauchy.

Incidental to his proofs of the Fundamental Theorem of Algebra (which might also be regarded as a part of our subject), Gauss showed that a polynomial $f(z) = z^n + A_1 z^{n-1} + \cdots + A_n$ has no zeros outside certain circles $|z| = R$. In the case that the A_j are all real, he showed in 1799 that $R = \max(1, 2^{1/2}S)$ where S is the sum of the positive A_j and he showed in 1816 that $R = \max(2^{1/2}\, n \mid A_k \mid)^{1/k}$, $k = 1, 2, \cdots, n$, whereas in the case of arbitrary, real or complex A_j, he showed in 1849 [Gauss 2] that R may be taken as the positive root of the equation $z^n - 2^{1/2}(|A_1| z^{n-1} + \cdots + |A_n|) = 0$. As a further indication of Gauss' interest in the location of the complex zeros of polynomials, we have his letter to Schumacher [Gauss 1, vol. X, pt. 1 p. 130, pt. 2 pp. 189-191] dated April 2, 1833, in which he tells of having written enough upon that topic to fill several volumes, but unfortunately the only results he subsequently published are those in Gauss [2]. The statement of his important result (our Th. (3,1)) on the mechanical interpretation of the zeros of the derivative of a polynomial comes to us only by way of a brief entry which he made presumably about 1836 in a notebook otherwise devoted to astronomy.

Cauchy also added much of value to our subject. About 1829 he derived for the moduli of the zeros of a polynomial more exact bounds than those given by Gauss. We shall describe these bounds in Sec. 27. To him we also owe the *Theory of Indices* (about 1837) as well as the even more fundamental *Principle of Argument.* (See Secs. 1 and 37.)

Since the days of Gauss and Cauchy, many other mathematicians have contributed to the further growth of the subject. In part this development resulted from the efforts to extend from the real domain to the complex domain the familiar theorems of Rolle, Descartes and Sturm. In part, also, it was stimulated by the discovery, in the general theory of functions of a complex variable, of such theorems as the Picard Theorem, theorems which had no previous counterpart in the domain of real variables. In view of the many as yet unsettled questions, our subject continues to be in an active state of development.

The subject has been partially surveyed in the addresses delivered before various learned societies by Curtiss [2], Van Vleck [4], Kempner [7], and Marden [9]. Parts of the subject have been treated in Loewy [1], in Pólya-Szegö [1, vol. 2, pp. 55-65, 242-252] and in Bauer-Bieberbach [1, pp. 187-192, 204-220].

The most comprehensive treatment to date has been Dieudonné [11], a seventy-one page monograph devoted exclusively to our subject.

Though very excellent, these surveys have been handicapped by a lack of the space required for an adequate treatment of the subject. There still remains the need for a detailed exposition which would bring together results at present scattered throughout the mathemetical journals and which would endeavor to unify and to simplify both the results and the methods of treatment.

The present book is an attempt to fill this need. In it an effort will be made to present the subject as completely as possible within the allotted space. Some of the results which could not be included in the main text have been listed as exercises, with occasional hints as to how they may be derived by use of the material in the main text. In addition, our bibliography refers each listed paper to the section of our text containing the material most closely allied to that in the paper, whether or not an actual reference to that paper is made in our text.

It is hoped that this book will serve the present and prospective specialist in the field by acquainting him with the current state of knowledge in the various phases of the subject and thus by helping him to avoid in the future the duplication of results which has occurred all too frequently in the past. It is hoped also that this book will serve the applied mathematician and engineer who need to know about the distribution of the zeros of polynomials when dealing with such matters as the formulation of stability criteria. Finally, it is hoped that this book will serve the general mathematical reader by introducing him to some relatively new, interesting and significant material of geometric nature—material which, though derived by essentially elementary methods, is not readily available elsewhere.

In closing, the author wishes to express his deep gratitude to Professor Joseph L. Walsh of Harvard University for having initiated the author into this field and for having encouraged his further development in it; also, for having made many helpful criticisms and suggestions concerning the present manuscript. The author wishes to acknowledge his indebtedness to The University of Wisconsin in Milwaukee for providing the assistance of Francis J. Stern in typing the manuscript and of Richard E. Barr, Jr. in drawing most of the accompanying figures; also his indebtedness to his colleagues at Madison for the opportunity of giving there, from February to June 1948, a course of lectures based upon the material in this book. Last but not least, the author wishes to thank the American Mathematical Society for granting him the privilege of publishing this manuscript in the Mathematical Surveys Series.

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MORRIS MARDEN.

ABBREVIATIONS

Eq. (m, n)	The equality given in the nth formula of section m.
Ineq. (m, n)	The inequality given in the nth formula of section m.
Ex. (m, n)	The nth exercise given at the close of section m.
Fig. (m, n)	The nth figure accompanying section m.
Th. (m, n)	The nth theorem in section m.
Cor. (m, n)	A corollary to Th. (m, n).
Lem. (m, n)	A lemma used in proving Th. (m, n). If several corollaries or lemmas go with Th. (m, n), they are distinguished by use of a letter written following the number n. Thus, Cor. (2, 3b) signifies the second corollary to Th. (2, 3).
$C(n, m)$	The binomial coefficient $n!/m!(n - m)!$.
sg x	Sign of real number x; 1, 0, or -1 according as $x > 0$, $x = 0$, or $x < 0$.
$\Re(z)$	Real part of the complex number $z = x + iy$.
$\Im(z)$	Imaginary part of the complex number $z = x + iy$.
arg z	Argument (amplitude, phase angle) of z.
$\bar{z}$	$x - iy$, conjugate imaginary of z.
$f^{(k)}(z)$	kth derivative of $f(z)$; exc. $f'(z), f''(z), f'''(z)$ for $k = 1, 2, 3$.
Smith [n] or [Smith n]	The nth article listed under the name of Smith in the bibliography.

CHAPTER I

INTRODUCTION

1. Some basic theorems. Before proceeding to the study of various specific problems connected with the zeros of polynomials, we shall find it useful to consider certain general theorems to which we shall make frequent reference.

The first of these theorems provides an intuitively obvious sufficient condition for the nonvanishing of a sum of complex numbers. It requires that each term in the sum be a vector drawn from the origin to a point on the same side of some line through the origin. This theorem may be stated as follows.

THEOREM (1,1). *If each complex number w_j, $j = 1, 2, \cdots, p$, has the properties that $w_j \neq 0$ and*

$$\gamma \leqq \arg w_j < \gamma + \pi, \qquad j = 1, 2, \cdots, p, \tag{1,1}$$

where γ is a real constant, then their sum $w = \sum_1^p w_j$ cannot vanish.

In proving Th. (1,1), we begin with the case $\gamma = 0$ when the w_j are vectors drawn from the origin to points on the positive axis of reals or in the upper half-plane. If $\arg w_j = 0$ for all j, then $\Re(w_j) > 0$ for all j and hence $\Re(w) > 0$. If $\arg w_j \neq 0$ for some value of j, then $\Im(w_j) > 0$ for that j and hence $\Im(w) > 0$. Thus, if $\gamma = 0$, $w \neq 0$.

In the case that $\gamma \neq 0$, we may consider the quantities $w_j' = e^{-\gamma i} w_j$. These satisfy ineq. (1,1) with $\gamma = 0$ and consequently their sum w' does not vanish. As $w' = e^{-\gamma i} w$, it follows that $w \neq 0$.

This proof establishes not merely that $w \neq 0$, but also the following. The point w lies inside the *convex sector* consisting of the origin and all the points z for which $\gamma \leqq \arg z \leqq \gamma + \delta$, $\delta < \pi$, if all the points w_j lie in the same sector.

Our second theorem expresses the so-called *Principle of Argument.*

THEOREM (1,2). *Let $f(z)$ be analytic interior to a simple closed Jordan curve C and continuous and different from zero on C. Let K be the curve described in the w-plane by the point $w = f(z)$ and let $\Delta_C \arg f(z)$ denote the net change in $\arg f(z)$ as the point z traverses C once over in the counterclockwise direction. Then the number p of zeros of $f(z)$ interior to C, counted with their multiplicities, is*

$$p = (1/2\pi)\, \Delta_C \arg f(z). \tag{1,2}$$

That is, it is the net number of times that K winds about the point $w = 0$.

We shall prove Th. (1,2) only in the case that $f(z)$ is a polynomial. If z_1, $z_2, \cdots, z_p$ denote the zeros of $f(z)$ inside C and $z_{p+1}, z_{p+2}, \cdots, z_n$ denote those outside C, then

$$f(z) = a_n(z - z_1) \cdots (z - z_p)(z - z_{p+1}) \cdots (z - z_n),$$

$$\arg f(z) = \arg a_n + \sum_{j=1}^{p} \arg (z - z_j) + \sum_{j=p+1}^{n} \arg (z - z_j).$$

As the point z describes C counterclockwise (see fig. (1,1)), arg $(z - z_j)$ increases by 2π when $1 \leqq j \leqq p$, but has a zero net change when $p < j \leqq n$. This fact leads at once to eq. (1,2).

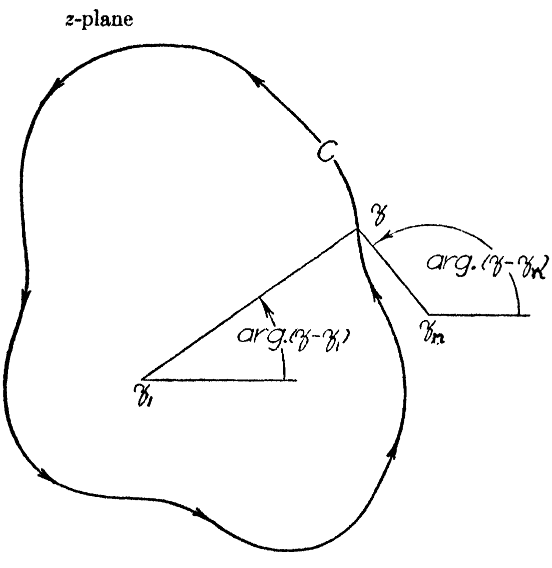

FIG. (1, 1)

As is well known, eq. (1,2) may be written as

$$p = \frac{1}{2\pi i} \int_C [f'(z)/f(z)]\, dz$$

when there is added to Th. (1,2) the hypothesis that C be a regular curve.

From Th. (1,2) we shall next derive the important

ROUCHÉ'S THEOREM. (Th. 1,3). *If $P(z)$ and $Q(z)$ are analytic interior to a simple closed Jordan curve C and if they are continuous on C and*

(1,3) $$|P(z)| < |Q(z)| \qquad \text{on} \qquad C,$$

then the function $F(z) = P(z) + Q(z)$ has the same number of zeros interior to C as does $Q(z)$. [Rouché 1].

For this purpose, we shall write

(1,4) $$F(z) = wQ(z), \qquad w = 1 + [P(z)/Q(z)].$$

If q denotes the number of zeros of $Q(z)$ in C, then according to Th. (1,2)

$$\Delta_C \arg Q(z) = 2\pi q. \tag{1,5}$$

Since $|P(z)/Q(z)| < 1$ on C, the point w defined in eqs. (1,4) describes (see fig. 1, 2) a closed curve Γ which lies interior to the circle with center at $w = 1$ and radius 1. Thus, point w remains always in the right half-plane. The net change in arg w as w varies on Γ is therefore zero. This means according to eqs. (1,4) and (1,5) that

$$\Delta_C \arg F(z) = \Delta_C \arg w + \Delta_C \arg Q(z) = 2\pi q$$

and according to Th. (1,2) that $F(z)$ has also q zeros in C.

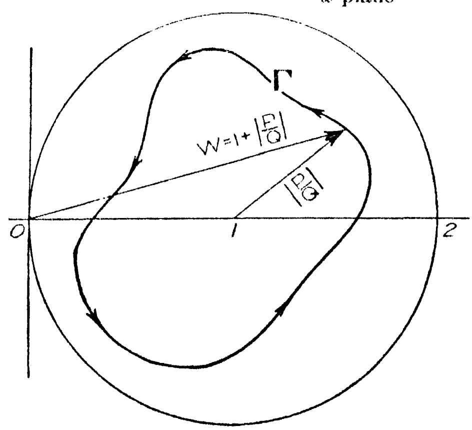

Fig. (1, 2)

We shall now apply Rouché's Theorem to a proposition which we shall often use either explicitly or implicitly. It is the proposition that the zeros of a polynomial are continuous functions of the coefficients of the polynomial. In more precise language, it may be stated as

Theorem (1,4). *Let*

$$f(z) = a_0 + a_1 z + \cdots + a_n z^n = a_n \prod_{j=1}^{p} (z - z_j)^{m_j}, \qquad a_n \neq 0,$$

$$F(z) = (a_0 + \epsilon_0) + (a_1 + \epsilon_1)z + \cdots + (a_{n-1} + \epsilon_{n-1})z^{n-1} + a_n z^n$$

and let

$$0 < r_k < \min |z_k - z_j|, \qquad j = 1, 2, \cdots, k-1, k+1, \cdots, p. \tag{1,6}$$

There exists a positive number ϵ such that, if $|\epsilon_i| \leqq \epsilon$ for $i = 0, 1, \cdots, n-1$, then $F(z)$ has precisely m_k zeros in the circle C_k with center at z_k and radius r_k.

To prove Th. (1,4), we have only to note (cf. Bieberbach-Bauer [1, p. 35]) that on C_k the polynomial

$$\zeta(z) = \epsilon_0 + \epsilon_1 z + \cdots + \epsilon_{n-1} z^{n-1}$$

has the property

$$| \zeta(z) | \leqq \epsilon M_k , \qquad M_k = \sum_{j=0}^{n-1} (r_k + | z_k |)^j ;$$

whereas on C_k

$$| f(z) | \geqq | a_n | r_k^{m_k} \prod_{j=1, j \neq k}^{p} (| z_j - z_k | - r_k)^{m_j} = \delta_k > 0.$$

If we choose $\epsilon < \delta_k / M_k$, we have the relation $| \zeta(z) | < | f(z) |$ on C_k. This means according to Rouché's Theorem that $F(z)$ has the same number of zeros in C_k as does $f(z)$. Since ineq. (1,6) ensures that the only zero of $f(z)$ in C_k is the one of multiplicity m_k at z_k, we see that $F(z)$ has precisely m_k zeros in C_k.

For other proofs of Th. (1,4) and similar theorems, the reader is referred to Weber [1], Coolidge [1], Maluski [1], Cippola [1], Krawtchouk [1], Waerden [1], Ostrowski [1, pp. 209-219], Kneser [1], and Iglisch [1].

Th. (1,4) may be regarded as a special case of the

HURWITZ THEOREM. (Th. 1,5). *Let $f_n(z)$ $(n = 1, 2, \cdots)$ be a sequence of functions which are analytic in a region R and which converge uniformly to a function $f(z) \not\equiv 0$ in every closed subregion of R. Let ζ be an interior point of R. If ζ is a limit point of the zeros of the $f_n(z)$, then ζ is a zero of $f(z)$. Conversely, if ζ is an m-fold zero of $f(z)$, every sufficiently small neighborhood K of ζ contains exactly m zeros (counted with their multiplicities) of each $f_n(z)$, $n \geqq N(K)$* [Hurwitz 1].

To prove Th. (1,5), let us first assume that $f(\zeta) \neq 0$. Since $f(z)$ is analytic in R, it can have only a finite number of zeros in R. We may then choose a positive ρ such that $f(z) \neq 0$ (in and) on the circle K: $| z - \zeta | = \rho$. Let us set $\epsilon = \min | f(z) |$ for z on K. Since the $f_n(z)$ converge to $f(z)$ uniformly in and on K, we can find a positive integer $N = N(K)$ such that $| f_n(z) - f(z) | < \epsilon$ for all z in and on K and all $n \geqq N$. Consequently, $| f_n(z) - f(z) | < | f(z) |$ on K and, by Rouché's Theorem, the sum function $f_n(z) = [f_n(z) - f(z)] + f(z)$ has as many zeros in K as does $f(z)$. Since therefore $f_n(z) \neq 0$ in K for all $n \geqq N$, a point ζ at which $f(\zeta) \neq 0$ cannot be a limit point of the zeros of the $f_n(z)$.

Conversely, if we assume that ζ is an m-fold zero of $f(z)$, we may again choose a positive ρ so that $f(z) \neq 0$ on K. Reasoning as in the previous paragraph, we now conclude from Rouché's Theorem that each $f_n(z)$, $n \geqq N$, has precisely m zeros in K.

Th. (1,5), whose proof we have now completed, will provide our principal means of passing from certain theorems on the zeros of polynomials to the corresponding theorems on the zeros of entire functions and perhaps of other analytic functions.

EXERCISES. Prove the following.

1. If each of p vectors w_j drawn from the origin lies in the closed half-plane $\gamma \leqq \arg w \leqq \gamma + \pi$ and if at least one of them lies in the open half-plane $\gamma < \arg w < \gamma + \pi$, then the sum $w = \sum_1^p w_j \neq 0$.

2. Th. (1,1) and ex. (1,1) hold for convergent infinite sums $\sum_1^\infty w_j$ in which all the w_j satisfy ineq. (1,1); also for integrals $\int_a^b w(t)\, dt$ in which a and b are real numbers and in which $w(t)$ is a continuous function of the real variable t and $\gamma \leqq \arg w(t) < \gamma + \pi$ for $a \leqq t \leqq b$.

3. Let $w = \sum_{j=1}^{p+1} w_j$. If p of the points w_j lie in the circle $|z| \leqq R_0$ and the remaining point w_j lies in the annulus $R_1 \leqq |z| \leqq R_2$ where $R_1 > pR_0$, then the point w lies in the annulus $R_1 - pR_0 \leqq |z| \leqq R_2 + pR_0$. Hence $w \neq 0$.

4. If the point z traverses a line L in a specified direction, then the net change in $\arg(z - z_1)$ is π or $-\pi$ according as z_1 is to the left or to the right of L relative to the specified direction.

5. THEOREM (1,6). *Let L be a line on which a given nth degree polynomial $f(z)$ has no zeros. Let $\Delta_L \arg f(z)$ denote the net change in $\arg f(z)$ as point z traverses L in a specified direction and let p and q denote the number of zeros of $f(z)$ to the left and to the right of this direction of L respectively. Then*

$$p - q = (1/\pi)\Delta_L \arg f(z) \tag{1,7}$$

and thus

$$p = (1/2)[n + (1/\pi)\Delta_L \arg f(z)], \tag{1,8}$$

$$q = (1/2)[n - (1/\pi)\Delta_L \arg f(z)]. \tag{1,9}$$

6. The polynomial $g(z) = z^n + b_1 z^{n-1} + \cdots + b_n$ has at least m zeros in an arbitrary neighborhood of the point $z = c$ if $|g^{(k)}(c)| \leqq \epsilon$ for $k = 0, 1, \cdots, m - 1$ and for ϵ a sufficiently small positive number [Kneser 1, Iglisch 1]. Hint: Use Rouché's Theorem.

7. Rouché's Theorem is valid when $|P(z)| \leqq |Q(z)|$ on C provided $F(z) \neq 0$ on C.

8. Rouché's Theorem is valid when C is the circle $|z| = 1$ and when $|P(z)| \leqq |Q(z)|$ on C, provided that at each zero Z of $F(z)$ on C the function $R(z) = \log Q(z)/P(z)$ has the properties $R'(Z) \neq 0$, $\Re(ZR'(Z)) < 0$, $\Im(ZR'(Z)) = 0$ [Lipka 3].

9. Let C be a closed Jordan curve inside which $P(z)$ and $Q(z)$ are analytic. On C let $P(z)$ and $Q(z)$ be continuous, $Q(z) \neq 0$ and $\Re[P(z)/Q(z)] > 0$. Then inside C, $P(z)$ has the same number of zeros as does $Q(z)$.

2. The zeros of the derivative. Mindful of the importance of Rolle's Theorem in the theory of functions of a real variable, we shall begin our detailed treatment

of the zeros of polynomials in a complex variable by studying the location of the zeros of the derivative $f'(z)$ of the polynomial

$$(2,1)\qquad f(z) = (z - z_1)^{m_1}(z - z_2)^{m_2} \cdots (z - z_p)^{m_p}, \qquad n = \sum_{j=1}^{p} m_j ,$$

in relation to the distinct zeros z_j of $f(z)$.

Since $f'(z)$ may be written as

$$f'(z) = f(z)[f'(z)/f(z)] = f(z)\, d\,[\log f(z)]/dz,$$

its zeros fall into two classes. First, there are the points z_j for which $m_j > 1$; as zeros of $f'(z)$, they have the individual multiplicities of $m_j - 1$ and the total multiplicity of $n - p$. Secondly, there are the $p - 1$ zeros of the logarithmic derivative

$$(2,2)\qquad F(z) = d\,[\log f(z)]/dz.$$

In most of our problems, we shall prescribe the location of the zeros of $f(z)$ and consequently we shall know *a priori* the location of the first class of zeros of $f'(z)$. It will remain for us to determine the location of the second class of zeros of $f'(z)$, namely those of $F(z)$. From eqs. (2,1) and (2,2), we see that these are the zeros of the function

$$(2,3)\qquad F(z) = \sum_{j=1}^{p} \frac{m_j}{z - z_j}$$

in which the m_j are positive integers.

In order to gain some insight into the problems about to be considered, we shall now interpret the zeros of $f'(z)$ from the standpoint of physics, geometry and function theory. Since our physical and geometrical interpretations will not use the fact that the m_j are positive integers, we shall express these interpretations as theorems concerning the zeros of a rational function $F(z) = g(z)/f(z)$ whose decomposition into partial fractions has the form (2,3) with the m_j as arbitrary real constants. In our function-theoretic interpretation, however, we shall find it convenient to restrict the m_j to be positive integers.

3. Physical interpretations. In place of $F(z)$, let us introduce its conjugate imaginary

$$(3,1)\qquad \overline{F}(\bar{z}) = \sum_{j=1}^{p} m_j w_j , \qquad w_j = 1/(\bar{z} - \bar{z}_j).$$

If we write $z - z_j = \rho_j e^{i\phi_j}$, then the jth term in eq. (3,1) is

$$m_j w_j = m_j (1/\rho_j) e^{i\phi_j}.$$

It may hence be represented by a vector having the direction from z_j to z and having the magnitude of m_j times the reciprocal of the distance from z_j to z. In other words, the jth term may be regarded as the force with which a fixed

mass (or electric charge) m_j at z_j repels (attracts if $m_j < 0$) a movable unit mass (or charge) at z, the law of repulsion being the inverse distance law.

An equivalent interpretation may be made in terms of masses repelling according to the inverse square law. For this purpose let us recall a result derived in books on Newtonian Potential Theory [O. D. Kellogg, *Potential Theory*, Berlin 1929, p. 10, ex. 5]. If an infinite, thin rod L_j of linear mass (or charge) density $m_j/2$ passes through the point z_j at right angles to the z-plane, the resultant force upon a unit particle at z due to the particles of L_j repelling according to the inverse-square law is a force directed along the line from z_j to z and inversely proportional to the distance from z_j to z. This means that alternatively we may interpret the conjugate imaginary of $F(z)$ as the resultant force at z, in the Newtonian field due to a system of n infinite thin rods (line charges) L_j at z_j, $j = 1, 2, \cdots, n$.

Still another interpretation is that $m_j w_j$ is the velocity vector in the two-dimensional flow of an incompressible fluid due to a source of strength m_j at z_j (sink if $m_j < 0$). Thus $\overline{F(\bar{z})}$ is the resultant velocity vector in a two-dimensional flow due to the systems of sources of strength m_j at the points z_j [L. M. Milne-Thomson, *Theoretical Hydrodynamics*, London 1938, p. 197].

Corresponding to each of these three physical interpretations of the function $F(z)$, we have a physical interpretation of the zeros of $F(z)$. In the first two cases, these zeros are the positions of equilibrium in the given force fields. In the third case, these zeros are the positions at which the velocity vanishes; that is, they are the so-called *stagnation points*.

We may summarize these results by stating two theorems. In the case that the m_j are positive integers, the first theorem is essentially due to Gauss [1], having been stated by him as a theorem on the zeros of the derivative of a polynomial. (Cf. our Preface.)

THEOREM (3,1). *The zeros of the function* $F(z) = \sum_1^p m_j/(z - z_j)$ *with all* m_j *real are the points of equilibrium in the field of force due to the system of* p *masses (point charges)* m_j *at the fixed points* z_j *repelling a movable unit mass at* z *according to the inverse distance law.*

THEOREM (3,2). *The zeros of* $F(z)$ *are the equilibrium points in the Newtonian field due to the system of* p *infinite, thin rods (line charges) of mass (or charge) density* $m_j/2$ *at the points* z_j. *They are also the stagnation points in the two-dimensional flow of an incompressible fluid due to* p *sources of strength* m_j *at the points* z_j.

4. Geometric interpretation. Let us begin with the case $p = 3$ when eq. (2,3) becomes

$$F(z) = \frac{m_1}{z - z_1} + \frac{m_2}{z - z_2} + \frac{m_3}{z - z_3}. \tag{4,1}$$

We note that $F(z)$ has two zeros z_1' and z_2', unless $n = m_1 + m_2 + m_3 = 0$ when

it has only one finite zero z_1'. We wish to know the location of the points z_1' and z_2' relative to the triangle with vertices z_1, z_2 and z_3.

The answer to this question is given by

THEOREM (4,1). *The zeros z_1' and z_2' of the function $F(z) = \sum_1^3 m_j(z - z_j)^{-1}$ are the foci of the conic which touches the line segments (z_1, z_2), (z_2, z_3) and (z_3, z_1) in the points ζ_3, ζ_1, and ζ_2 that divide these segments in the ratios $m_1 : m_2$, $m_2 : m_3$ and $m_3 : m_1$, respectively. If $n = m_1 + m_2 + m_3 \neq 0$, the conic is an ellipse or hyperbola according as $nm_1m_2m_3 > 0$ or < 0; whereas, if $n = 0$ but $\nu = m_1z_1 + m_2z_2 + m_3z_3 \neq 0$, the conic is a parabola whose axis is parallel to the line joining the origin to point ν.*

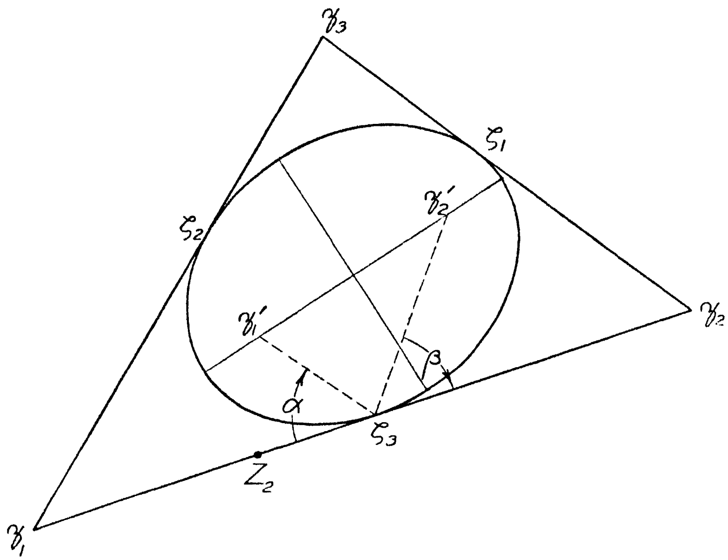

FIG. (4, 1)

To prove this theorem in the case leading to the ellipse (see fig. 4,1), let us make use of a characteristic property of the ellipse. At any point P of an ellipse the focal radii should lie to the same side of the tangent and make equal angles with this tangent.

Let us consider the line (z_1, z_2) and the point ζ_3 on this line. By definition,

$$\zeta_3 = (m_1z_2 + m_2z_1)/(m_1 + m_2); \tag{4,2}$$

that is,

$$\frac{m_1}{\zeta_3 - z_1} + \frac{m_2}{\zeta_3 - z_2} = 0. \tag{4,3}$$

We may on combining the fractions in eq. (4,1) write

(4,4) $$F(z) = \frac{n(z - z_1')(z - z_2')}{(z - z_1)(z - z_2)(z - z_3)}.$$

Thus from eqs. (4,1), (4,3) and (4,4), we find

$$F(\zeta_3) = \frac{m_3}{\zeta_3 - z_3} = \frac{n(\zeta_3 - z_1')(\zeta_3 - z_2')}{(\zeta_3 - z_1)(\zeta_3 - z_2)(\zeta_3 - z_3)}.$$

That is,

(4,5) $$\frac{(z_1' - \zeta_3)(z_2' - \zeta_3)}{(z_1 - \zeta_3)(z_2 - \zeta_3)} = \frac{m_3}{n}.$$

In the case $m_1m_2 > 0$, we observe that $nm_3 > 0$ and that ζ_3 is an internal division point of the line segment z_1z_2. From eq. (4,5) we obtain the result:

(4,6) $$\arg\left(\frac{z_1' - \zeta_3}{z_1 - \zeta_3}\right) - \arg\left(\frac{z_2 - \zeta_3}{z_2' - \zeta_3}\right) = \arg\frac{n}{m_3} = 0.$$

As shown in fig. (4,1) this means that z_1' and z_2' must both lie on the same side of line z_1z_2 and that angle α = angle β.

In the case $m_1m_2 < 0$, we observe that $nm_3 < 0$ and that ζ_3 is an external division point of the line segment z_1z_2. From (4,5) we have the relation

(4,7) $$\arg\frac{z_1' - \zeta_3}{z_1 - \zeta_3} - \arg\frac{(-z_2 + \zeta_3)}{(z_2' - \zeta_3)} = \arg\left(-\frac{n}{m_3}\right) = 0.$$

This means as shown in fig. (4,1) with z_2 taken at point Z_2, that z_1' and z_2' must both be on the same side of line z_1z_2 and that angle α = angle β.

In all cases therefore, in which $nm_1m_2m_3 > 0$, the points z_1' and z_2' are the foci of the ellipse described in Th. (4,1). The proof of Th. (4,1) in the remaining cases is left to the reader.

The theorem just established is a special case of the following:

THEOREM (4,2). *The zeros of the function*

(4,8) $$F(z) = \sum_{j=1}^{p} \frac{m_j}{z - z_j}, \qquad m_j \text{ real, } m_j \neq 0,$$

are the foci of the curve of class $p - 1$ which touches each line-segment z_jz_k in a point dividing the line segment in the ratio $m_j : m_k$.

The proof of Theorem (4,2) is necessarily less elementary than that of Theorem (4,1). The one which follows will make use of line-co-ordinates and some abridged notation.

We may write eq. (4,8) in the form

(4,9) $$\sum_{j=1}^{p} \frac{m_j}{tx_j + ity_j - 1} = 0, \qquad t = 1/(x + iy).$$

Let us compare (4,9) with the equation

$$\sum_{j=1}^{p} \frac{m_j}{\mathfrak{L}_j} = 0, \qquad \mathfrak{L}_j = \lambda x_j + \mu y_j - 1. \tag{4,10}$$

When cleared of fractions this equation has the form

$$\Phi(\mu, \lambda) = m_1\mathfrak{L}_2\mathfrak{L}_3 \cdots \mathfrak{L}_p + m_2\mathfrak{L}_1\mathfrak{L}_3 \cdots \mathfrak{L}_p + \cdots + m_p\mathfrak{L}_1\mathfrak{L}_2 \cdots \mathfrak{L}_{p-1} = 0 \tag{4,11}$$

and hence represents a curve C of class $p - 1$. Eq. (4,9) tells us that eq. (4,10) is satisfied by the line with the co-ordinates

$$\lambda = 1/(x + iy), \qquad \mu = i/(x + iy), \tag{4,12}$$

a line which, consequently, is tangent to curve C. Since line (4,12) is an isotropic line through point (x, y), point (x, y) must be a focus of curve C. Furthermore, the line (λ_0, μ_0) joining the two points (x_1, y_1), (x_2, y_2) satisfies simultaneously the two equations $\mathfrak{L}_1 = 0$ and $\mathfrak{L}_2 = 0$; viz.,

$$\lambda_0 x_1 + \mu_0 y_1 - 1 = 0, \qquad \lambda_0 x_2 + \mu_0 y_2 - 1 = 0. \tag{4,13}$$

That is, it satisfies eq. (4,11) and is hence tangent to curve (4,11).

Now the point of contact of a tangent line (λ_0, μ_0) has the line equation

$$(\lambda - \lambda_0)\left(\frac{\partial \Phi}{\partial \lambda}\right)_0 + (\mu - \mu_0)\left(\frac{\partial \Phi}{\partial \mu}\right)_0 = 0, \tag{4,14}$$

where the subscript 0 indicates values at (λ_0, μ_0). In view of equation (4,11),

$$\begin{aligned} \left(\frac{\partial \Phi}{\partial \lambda}\right)_0 &= (\mathfrak{L}_3\mathfrak{L}_4 \cdots \mathfrak{L}_p)_0[m_1x_2 + m_2x_1], \\ \left(\frac{\partial \Phi}{\partial \mu}\right)_0 &= (\mathfrak{L}_3\mathfrak{L}_4 \cdots \mathfrak{L}_p)_0[m_1y_2 + m_2y_1]. \end{aligned} \tag{4,15}$$

On discarding the common factor in eqs. (4,15), we may write (4,14) as

$$\begin{aligned} &[\lambda(m_1x_2 + m_2x_1) + \mu(m_1y_2 + m_2y_1)] \\ &\qquad - [\lambda_0(m_1x_2 + m_2x_1) + \mu_0(m_1y_2 + m_2y_1)] = 0. \end{aligned} \tag{4,16}$$

According to eqs. (4,13), the second bracket in (4,16) has the value $(m_1 + m_2)$ and thus (4,16) may be written in the form

$$m_2\mathfrak{L}_1 + m_1\mathfrak{L}_2 = 0. \tag{4,17}$$

If $m_1 + m_2 \neq 0$, this clearly is a line equation for the point

$$\left(\frac{m_2x_1 + m_1x_2}{m_2 + m_1}, \quad \frac{m_2y_1 + m_1y_2}{m_2 + m_1}\right).$$

Hence, the line-segment z_1z_2 has the desired properties.

In a like manner the same may be shown concerning the other line-segments z_jz_k, thus completing the proof of Th. (4,2).

Th. (4,2) was proved first by Siebeck [1] and later by Van den Berg [1], Vries [1], Juhel-Rénoy [1], Heawood [1], Occhipinti [1], Fujiwara [2], Linfield [1] and Haensel [1]. A proof covering only the special case $p = 3$, that is Th. (4,1), was given by Bôcher [2] and Grace [1] for the subcase $m_j > 0$, all j, and by Marden [13] for arbitrary m_j. Furthermore, Th. (4,2) has been extended to the kth derivative of a rational function by Fujiwara [2] and Linfield [1] and to certain entire functions by Reutter [1]. Also, in the case $p = 3$ and $m_1 = m_2 = m_3 = 1$, the result has been applied by Walsh [4] to the ruler-and-compass construction of the zeros of the derivative of a cubic polynomial.

5. **Function-theoretic interpretations.** Finally, for a polynomial

$$f(z) = (z - z_1)(z - z_2) \cdots (z - z_n) \tag{5,1}$$

let us interpret the q distinct zeros z'_j of its derivative

$$f'(z) = n(z - z'_1)^{p_1}(z - z'_2)^{p_2} \cdots (z - z'_q)^{p_q}, \qquad \sum_1^q p_j = n - 1, \tag{5,2}$$

from the point of view that $f(z)$ is an analytic function of the complex variable z. This means, as is well known, that $w = f(z)$ maps any finite region R of the z-plane upon a finite region S of the w-plane, the map being conformal except at the q points z'_j. Specifically, if two curves of the z-plane intersect at z'_j at an angle of ψ, they map into two curves of the w-plane that intersect at an angle of $(p_j + 1)\psi$. For this reason the zeros of $f'(z)$ are called the *critical points* of $f(z)$.

Let us consider, in particular, the families of curves

$$\Re f(z) = a, \tag{5,3}$$

$$\Im f(z) = b, \tag{5,4}$$

$$|f(z)| = c, \tag{5,5}$$

$$\arg f(z) = d, \tag{5,6}$$

where a, b, c $(c > 0)$ and d are real constants. These curves map into the families of curves in the w-plane $\Re(w) = a$, $\Im(w) = b$, $|w| = c$ and $\arg w = d$, respectively. Curves (5,3) to (5,6) have in common the fact that their multiple points occur at and only at the zeros z'_j of $f'(z)$.

For example, if we set $w = u + iv$, and thus write (5,5) as

$$F(x, y) = u^2 + v^2 - c^2 = 0,$$

we may find the multiple points of curve (5,5) as the common solutions of the equations

$$\partial F/\partial x = 2u(\partial u/\partial x) + 2v(\partial v/\partial x) = \Re(f'\bar{f}) = 0,$$

$$\partial F/\partial y = 2u(\partial u/\partial y) + 2v(\partial v/\partial y) = -\Im(f'\bar{f}) = 0.$$

We infer from these equations, since $f(z) \neq 0$ on the curve (5,5), that every multiple point of this curve is a zero of $f'(z)$.

We shall now state briefly, largely without proof, some additional facts concerning the above families of curves.

Concerning the curves (5,3) and (5,4), the specimens $a = 0$ and $b = 0$ are of interest to us as the curves whose asymptotic behavior is the basis of Gauss' fourth proof of the Fundamental Theorem of Algebra [Gauss 2; cf. Motzkin and Ostrowski 1]. Also, using these curves, Liouville [1] has given us a generalization of Rolle's Theorem, namely, that the arc z_1z_2 of the curve $\mathfrak{J}(f) = 0$ joining

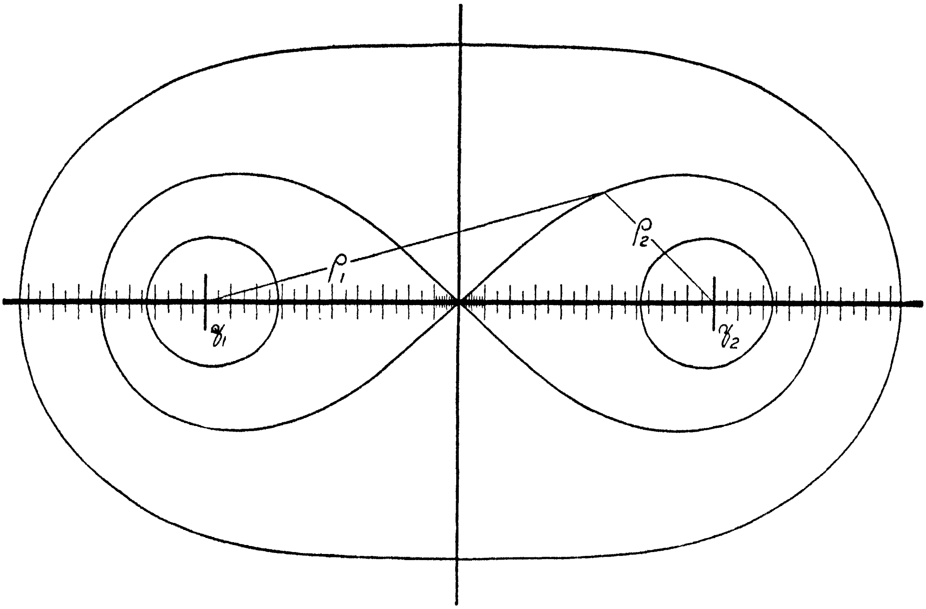

FIG. (5, 1)

a pair z_1 and z_2 of zeros of $f(z)$ passes through at least one zero z_j' of $f'(z)$. Furthermore, by considering the Riemann surface for $w = f(z)$, De Boer [1] was able to show that any k-branched curve (5,3) or (5,4) not passing through a point z_j' divides the plane into $k + 1$ regions each containing one less zero of $f'(z)$ than of $f(z)$.

Of still greater interest are the curves (5,5). From eqs. (5,1) and (5,5), we see that each is the locus of a point z which moves so that the product of its distances from the points z_j is the constant c. As such, each is called a *lemniscate* and the points z_j are called its poles. When $n = 1$, the lemniscate is a circle and, when $n = 2$, it is a Cassinian curve, as illustrated in fig. (5,1) for $z_1 = -1$ and $z_2 = 1$ with $c = 1/2$ (two small ovals), $c = 1$ (Bernoulli lemniscate) and $c = 2$ (single large oval).

Assuming now the points z_j to be distinct, let us describe the change in the

lemniscate (5,5) as we allow c to increase from 0 to ∞. For c very small, we infer from Th. (1,4) that the lemniscate consists of n small ovals each containing a point z_i. For any value of c, curve (5,5) consists of N closed Jordan curves $J_1, J_2, \cdots, J_N$, $N \leqq n$, the number N being a non-increasing function of c. Among these curves, no J_i encloses a J_k, $i \neq k$. If $c = |f(z_j')|$, $j = 1, 2, \cdots,$ or q, the curve (5,5) passes through the zero z_j' of $f'(z)$ and consequently has a multiple point at z_j', the $p_j + 1$ tangent lines at z_j' being equally spaced. If, however, $c < |f(z_j')|$ for $j = 1, 2, \cdots, k \leqq q$ and $c > |f(z_j')|$ for $j = k + 1, k + 2, \cdots, q$; that is, if exactly k of the points z_j' lie exterior to the lemniscate and the remaining z_j' lie interior to the curve, then the lemniscate consists of

$$N = 1 + p_1 + p_2 + \cdots + p_k$$

Jordan curves J_i, each one of which contains one less zero z_j' of $f'(z)$ than zeros z_j of $f(z)$, all counted with their multiplicities.

As Walsh [16] has shown, the use of the lemniscates (5,5) and of the properties of the zeros of $f'(z)$ provides a method of determining the location of the critical points of the Green's function of a region R. For example, if R is an infinite region which has a finite boundary B and for which $G(x, y)$, the Green's function with pole at infinity, exists, then we may select a sequence $L_1, L_2, L_3, \cdots$ of lemniscates approaching B monotonically. If the equation of L_k is $|f_k(z)| = c_k$, the Green's function for the exterior of L_k is

$$G_k(x, y) = (1/n_k) \log (|f_k(z)|/c_k),$$

where n_k is the degree of the polynomial $f_k(z)$. The function $G_k(x, y)$ is essentially the potential due to particles (or charges) at the zeros of $f_k(z)$, and the lemniscates (5,5) are the equipotential curves in the force-fields described in sec. 3. The critical points of $G_k(x, y)$ are clearly the zeros z_{kj}' of $f_k'(z)$. If we now apply Hurwitz' Theorem (Th. 1,5), we may obtain the critical points of $G(x, y)$ as the limit points of the z_{kj}'.

The reader is referred to Walsh [16] for a detailed treatment of the lemniscates (5,5) and their application to the critical points of harmonic functions. For older treatments of lemniscates the reader is referred to Loria [1], Lucas [2], Stieltjes [1], Mansion [1], MacDonald [1], Poussin [1], Dall'Agnola [1] and Lange-Nielsen [1]. Finally, for a treatment of the critical points of harmonic functions directly (without a limiting process) as equilibrium points in a force-field, the reader is referred to Walsh [18] and [20].

CHAPTER II

THE CRITICAL POINTS OF A POLYNOMIAL AND SOME OF THEIR GENERALIZATIONS

6. Their convex enclosure. In the previous chapter, we found that any critical point (zero of the derivative) of the polynomial

$$(6,1) \qquad f(z) = (z - z_1)^{m_1}(z - z_2)^{m_2} \cdots (z - z_p)^{m_p},$$

if not a multiple zero of $f(z)$, is a zero of the function

$$(6,2) \qquad F(z) = \sum_{j=1}^{p} \frac{m_j}{z - z_j}.$$

We found also that the zeros of $F(z)$ can be interpreted in various ways from the standpoint of physics, geometry and function theory. In the present chapter we shall employ these interpretations and some additional analysis to determine the relative positions of the zeros of $F(z)$ and of the points z_j. We shall also, by the same analysis, determine the location of the zeros of rational functions of a more general form than (6,2), as well as the zeros of certain systems of functions of a form similar to (6,2).

The relative position of the real zeros and critical points of a real differentiable function is described in the well-known Theorem of Rolle that between any two zeros of the function lies at least one zero of its derivative. However, Rolle's Theorem is not generally true for analytic functions of a complex variable. For example, the function $f(z) = e^{2\pi z i} - 1$ vanishes for $z = 0$ and $z = 1$, but its derivative $f'(z) = 2\pi i e^{2\pi i z}$ never vanishes. This leads to the question as to what generalizations or analogues of Rolle's Theorem are valid for at least a suitably restricted class of analytic functions, such as the polynomials in a complex variable. [Cf. Dieudonné 1].

In the present section we shall answer this question, not with respect to Rolle's Theorem, but rather with respect to a particular corollary of Rolle's Theorem. This says that any interval of the real axis which contains all the zeros of a polynomial $f(z)$ also contains all the zeros of the derivative $f'(z)$. This corollary may be replaced (see ex. (10,1)) by the more general theorem that *a line-segment L* (not necessarily on the real axis) *which contains all the zeros of a polynomial $f(z)$ also contains all the zeros of its derivative.* But this theorem is only a special case of the following result proved in 1874 by Lucas [1, 2, 3] and subsequently by Legebeke [1], De Boer [1], Berlothy [1], Cesàro [1], Bôcher [2], Grace [1], Hayashi [3], Irwin [1], Gonggryp [1], Porter [1], Uchida [1], Krawtchouck [2] and Nagy [1 and 5].

LUCAS' THEOREM. (Th. 6,1). *Any convex polygon K which contains all the zeros of a polynomial $f(z)$ also contains all the zeros of the derivative $f'(z)$.*

From a physical point of view, this theorem is an obvious consequence of Gauss' Theorem (Th. 3,1) with the m_i as positive integers. For, if the zeros (see fig. 6,1) of $f'(z)$ are either multiple zeros of $f(z)$ or the positions of equilibrium in the field of force due to masses at the zeros of $f(z)$, then in either case the zeros of $f'(z)$ must lie in or on any convex polygon enclosing the zeros of $f(z)$.

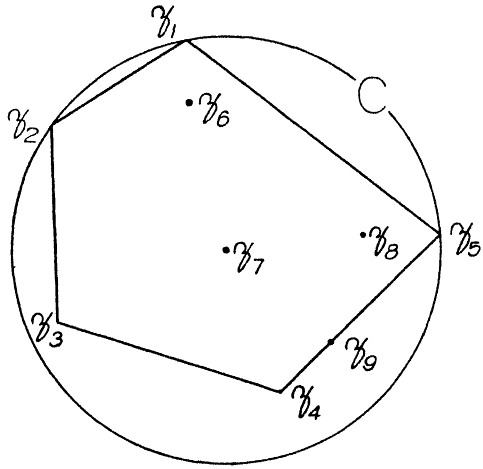

FIG. (6, 1)

To prove the theorem analytically, let us apply Th. (1,1). If z', a zero of $f'(z)$, were exterior to K, it could not be a multiple zero of $f(z)$. Furthermore, the angle subtended at z' by K would be $A(z')$, where $0 < A(z') < \pi$. Hence, if drawn from z, each of the vectors $(-w_i)$ of formula (3,1) would lie in $A(z')$ as would therefore each of the vectors $W_i = -m_iw_i$. Hence by Th. (1,1), $\overline{F(z')} = -(W_1 + W_2 + \cdots + W_p) \neq 0$. As this contradicts our assumption that z' is a zero of $f'(z)$, no zero of $f'(z)$ may lie exterior to polygon K.

From Th. (6,1) we may infer

THEOREM (6,2). *Any circle C which encloses all the zeros of a polynomial $f(z)$ also encloses all the zeros of its derivative $f'(z)$.*

For, if K is the smallest convex polygon enclosing the zeros of $f(z)$, then K lies in C and therefore by Theorem (6,1) all the zeros of $f'(z)$, being in K, also lie in C.

Conversely, Th. (6,1) follows from Th. (6,2). For, if Theorem (6,2) were valid, through each pair of vertices of the polygon K of Theorem (6,1) we could draw a circle C_k which contains K and hence the zeros of both $f(z)$ and $f'(z)$. The region K' common to all these circles C_k would also contain all the zeros of $f(z)$ and $f'(z)$. Since this holds for all choices of circles C_k passing through pairs of vertices of K and containing K, all the zeros of $f'(z)$ must lie in the region common to all possible regions K', that is, in the polygon K.

Thus, as stated in Walsh [2(a)], Ths. (6,1) and (6,2) are actually equivalent. Furthermore, they are the best possible theorems in the sense that, if the zeros of an nth degree polynomial $f(z)$, $n > 1$, are allowed to vary independently in and on a convex polygon K or circle C, every point of K or C is a possible multiple point of $f(z)$ and therefore a possible zero of $f'(z)$. If, however, the zeros of $f(z)$ are fixed, no zero of $f'(z)$ other than a multiple zero of $f(z)$ may lie on the polygon K or circle C or may lie in a certain neighborhood of each zero of $f(z)$. Cf. ex. (6,1) and ex. (26,5).

EXERCISES. Prove the following.

1. If the polynomial $f(z)$ has no multiple zeros, all the zeros of $f'(z)$ must lie interior to the polygon of Th. (6,1).

2. The zeros of $f^{(k)}(z)$, $1 \leqq k \leqq n - 1$, also lie in the polygon of Th. (6,1) and in the circle of Th. (6,2).

3. Any infinite convex region which contains all the zeros of an entire function $f(z)$ of genre zero also contains all the zeros of $f'(z)$. Hint: By definition $f(z) = \prod_{j=1}^{\infty} (1 - z/a_j)$. Use Ths. (1,5) and (6,1) [Porter 1].

4. If r is the smallest number such that all zeros of $f'(z)$ lie in $|z| \leqq r$, then at least one zero of $f(z)$ lies in $|z| \geqq r$. Hint: Use Th. (6,2).

5. Th. (6,2) is valid if the words "circle C" are replaced by "convex region K" [Lucas 3, p. 18].

6. Let $f(z) = c_k z^k + c_{k+1} z^{k+1} + \cdots + c_n z^n$, $c_k c_n \neq 0$, have all its zeros in a half-plane bounded by a line L through the origin, not all the zeros of $f(z)$ being on L. Then $c_j \neq 0$ for $k \leqq j \leqq n$ [Laguerre 1c, Weisner 3]. Hint: $c_j = f^{(j)}(0)/j! \neq 0$ by Lucas' Theorem.

7. **The critical points of a real polynomial.** In the Lucas Theorem (6,1) we treated the zeros z_j of $f(z)$ as independent parameters. Obviously, if we impose some mutual restraints upon the z_j, such as the requirement that the z_j be symmetrical in a line or point, we may expect the locus of the zeros of $f'(z)$ to be a smaller region than that given by the Lucas Theorem.

Let us in particular assume that $f(z)$ is a real polynomial and thus that its non-real zeros occur in conjugate imaginary pairs. Let us construct the circles whose diameters are the line-segments joining the pairs of conjugate imaginary zeros of $f(z)$. These circles we shall call the *Jensen circles of* $f(z)$. (See fig. 7,1).

We shall now state a theorem which was announced without proof by Jensen [1] in 1913. It was proved by Walsh [4] in 1920 and later by Echols [1] and Nagy [3].

JENSEN'S THEOREM. (Th. 7,1). *Every non-real zero of the derivative of a real polynomial $f(z)$ lies in or on at least one of the Jensen circles of $f(z)$.*

To establish this theorem, we note that in the equation

$$f'(z)/f(z) = \sum^{n} [1/(z - z_j)]$$

the sum of the terms $w_1 = 1/(x + iy - x_1 - iy_1)$ and $w_2 = 1/(x + iy - x_1 + iy_1)$ corresponding to the pair of zeros $z_1 = x_1 + iy_1$ and $z_2 = x_1 - iy_1$ has the imaginary part

$$\Im(w_1 + w_2) = \frac{-2y[(x - x_1)^2 + y^2 - y_1^2]}{[(x - x_1)^2 + (y - y_1)^2][(x - x_1)^2 + (y + y_1)^2]}$$

whereas the term $w_3 = 1/(x + iy - x_3)$ corresponding to a real zero $z_3 = x_3$ of $f(z)$ has the imaginary part

$$\Im(w_3) = -y/[(x - x_3)^2 + y^2].$$

Thus, sg $\Im(w_1 + w_2) = -$sg y for every point z outside all the Jensen circles and sg $\Im(w_3) = -$sg y for every point z. In other words, outside all the Jensen circles

$$\text{sg}\, \Im[f'(z)/f(z)] = -\,\text{sg}\, y. \tag{7,1}$$

In particular, if z is a non-real point outside all the Jensen circles, $f'(z) \neq 0$, a result which proves the Jensen Theorem.

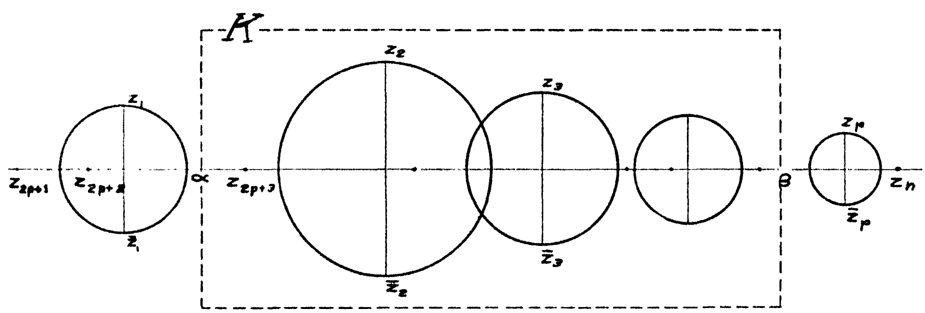

FIG. (7, 1)

Incidentally, we see that a non-real point on a Jensen circle cannot be a zero of $f(z)$ if it lies outside all the other Jensen circles.

The Jensen Theorem supplements Rolle's Theorem in describing the location of the zeros of $f'(z)$ relative to those of $f(z)$. A theorem which describes the number of zeros of $f'(z)$ is the following one due to Walsh [4].

THEOREM (7,2). *Let I: $\alpha \leqq x \leqq \beta$ be an interval of the real axis such that neither α nor β is a zero of the real polynomial $f(z)$ or is a point in or on any Jensen circle of $f(z)$. Let K be the configuration consisting of I and of the closed interiors of all the Jensen circles which intersect I. Then, if K contains k zeros of $f(z)$, it contains at least $k - 1$ and at most $k + 1$ zeros of $f'(z)$.*

We shall prove this theorem with the aid of Th. (1,2), the Principle of Argument. Let us denote by R the boundary of the smallest rectangle which has

sides parallel to the co-ordinate axes and which encloses K. In view of eq. (7,1) R is mapped by the function $w = f'(z)/f(z)$ upon the w-plane into a curve which encircles the origin at most once. Hence, $\Delta_R \arg [f'(z)/f(z)]$ is 0 or $\pm 2\pi$ and by eq. (1,2) the number of zeros of $f'(z)$ within R differs by at most one from the number of zeros of $f(z)$ in R.

An immediate consequence of Th. (7,2) is the following result also due to Walsh [4].

COROLLARY (7,2). *Any closed interval of the real axis contains at most one zero of $f'(z)$ if it contains no zero of $f(z)$ and if it is exterior to all the Jensen circles for $f(z)$.*

EXERCISES. Prove the following.

1. If a is a real constant and if $f(z)$ is a real polynomial whose derivative is $f'(z)$, none of the imaginary zeros of $F_1(z) = (D + a)f(z) = f'(z) + af(z)$ lies outside the Jensen circles of $f(z)$. Hint: Study the imaginary part of $a + f'(z)/f(z)$. [Jensen 1, Nagy 3].

2. Let $E_m(A, \bar{A})$ denote the ellipse having as minor axis the line-segment joining the pair of conjugate imaginary points A and $\bar{A}$ and as major axis a line-segment $m^{1/2}$ times as long as the minor axis. Then the envelope of the circles whose diameters are the vertical chords of $E_m(A, \bar{A})$ is the ellipse $E_{m+1}(A, \bar{A})$.

3. If a and b are real constants and $f(z)$ is a real polynomial whose first two derivatives are $f'(z)$ and $f''(z)$, then none of the imaginary zeros of

$$F_2(z) = f''(z) + (a + b)f'(z) + abf(z)$$

lies outside the ellipses having as minor axes the lines joining the pairs of conjugate imaginary zeros of $f(z)$ and having major axes $2^{1/2}$ times as large as the minor axes. Hint: Noting that $F_2(z) = (D + b)(D + a)f(z) = (D + b)F_1(z)$, apply twice the results of exs. 1 and 2 [Jensen 1, Nagy 3].

4. If $f(z)$ is a real polynomial and $g(z)$ an mth degree polynomial with only real zeros, then the non-real zeros of the polynomial

$$F_m(z) = g(D)f(z), \qquad D = d/dz,$$

lie in the ellipses which have as minor axes the lines joining the pairs of conjugate imaginary zeros of $f(z)$ and which have major axes $m^{1/2}$ times as long as their minor axes [Jensen 1, Nagy 3].

5. If $f(z)$ is any polynomial whose zeros are symmetric in the origin, then

(a) all the zeros of $f'(z)$ lie in any double sector $|\arg(\pm z)| < \gamma < \pi/4$ containing the zeros of $f(z)$;

(b) all the zeros of $f'(z)$ (except perhaps one at the origin) lie inside, outside or on any equilateral hyperbola H with center at 0 according as all the zeros of $f(z)$ also lie inside, outside or on H. Hint: By hypothesis $f(z) = z^k\phi(z^2)$. Show that the zeros of $F(W) = [f(W^{1/2})]^2$ lie in a convex point set, which by the Lucas Theorem must contain the zeros of $F'(W)$ [Walsh 13].

8. Some generalizations. From the proofs given in the last two sections, it is clear that the Lucas and Jensen Theorems are essentially results regarding the zeros of the function

$$F(z) = \sum_{j=1}^{n} m_j/(z - z_j), \qquad m_j > 0,$$

and that these results are valid even when the positive numbers m_j are not

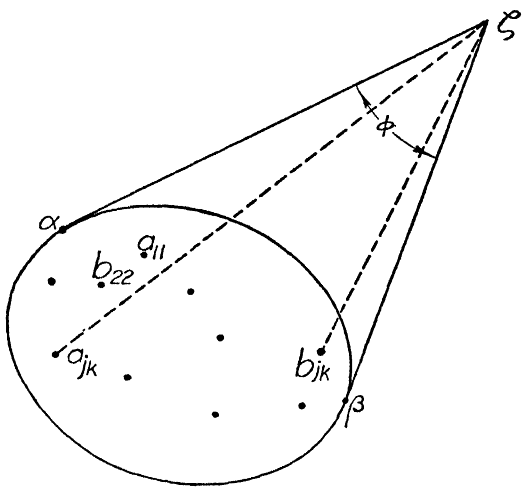

Fig. (8, 1)

integers. This expression is however only a special case of the linear combination

$$(8,1) \qquad F(z) = \sum_{j=1}^{n} m_j f_j(z)$$

where

$$f_j(z) = \frac{(z - a_{j1})(z - a_{j2}) \cdots (z - a_{jp})}{(z - b_{j1})(z - b_{j2}) \cdots (z - b_{jq})}$$

and where the m_j are complex numbers such that

$$(8,2) \qquad \mu \leqq \arg m_j \leqq \mu + \gamma < \mu + \pi, \qquad j = 1, 2, \cdots, n.$$

We ask now whether or not the Lucas Theorem (Th. 6,1) may be generalized to functions $F(z)$ of type (8,1).

We shall first prove

Theorem (8,1). *If K is a convex region which encloses all the zeros a_{jk} and poles b_{jk} of each $f_j(z)$ of eq. (8,1), then $F(\zeta) \neq 0$ at any point ζ at which K subtends an angle less than $\phi = (\pi - \gamma)/(p + q)$.*

Since ζ is necessarily exterior to K, we may find in K two points α and β such that (see fig. (8,1) where, however, K subtends at ζ the angle ϕ)

$$0 < \arg (\zeta - \beta)/(\zeta - \alpha) < \phi$$

and for all j and k

$$(8,3) \qquad 0 < \arg \sigma_{jk} < \phi, \qquad 0 < \arg \tau_{jk} < \phi,$$

where $\sigma_{jk} = (\zeta - a_{jk})/(\zeta - \alpha)$ and $\tau_{jk} = (\zeta - \beta)/(\zeta - b_{jk})$. Let us now set

$$(8,4) \qquad w_j = m_j f_j(z)[(\zeta - \beta)^q/(\zeta - \alpha)^p].$$

Since $w_j = m_j \prod_{k=1}^{p} \sigma_{jk} \prod_{k=1}^{q} \tau_{jk}$, we may, on use of eqs. (8,2) and (8,3), obtain the inequality

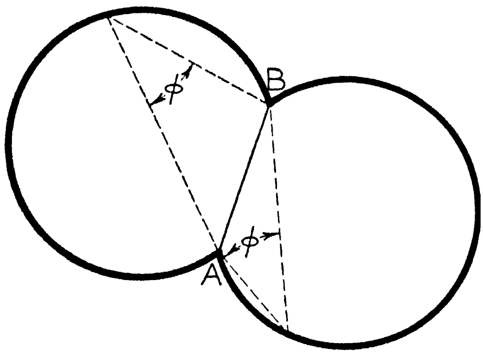

Fig. (8, 2)

$$\mu \leqq \arg w_j < \mu + \gamma + (p + q)\phi = \mu + \pi.$$

It follows now from Th. (1,1) and eqs. (8,1) and (8,4) that

$$F(\zeta)[(\zeta - \beta)^q/(\zeta - \alpha)^p] = \sum_{j=1}^{n} w_j \neq 0,$$

as required in Theorem (8,1).

If we define 2π to be the angle subtended by K at a point interior to K, we may say that the zeros of $F(z)$ lie in the region $S(K, \phi)$ comprised of all points at which K subtends an angle of at least ϕ. It is important therefore that we determine the nature of the region $S(K, \phi)$.

For example, if K is a circle of radius r, then $S(K, \phi)$ is a concentric circle of radius $r \csc (\phi/2)$. If K is an ellipse, then $S(K, \phi)$ is an oval-shaped region bounded by a fourth-order curve. If K is the line segment AB in fig. (8,2), then $S(K, \phi)$ will be bounded by two arcs of circles which pass through A and B and are symmetric in the line AB. If K is the closed interior of the triangle ABC in fig. (8,3), then $S(K, \phi)$ will be a triangular figure bounded by three circular arcs through the pairs of vertices A, B, C.

As the last two examples show, the region $S(K, \phi)$ is not in general a convex region, though it always contains K and coincides with K when $\phi = \pi$. The region $S(K, \phi)$ is, however, always *star-shaped* with respect to K. That is, it has the property that, if P is any point of K and if Q is any point of $S(K, \phi)$, then the entire line-segment PQ lies in $S(K, \phi)$. (Cf. fig. 8,3).

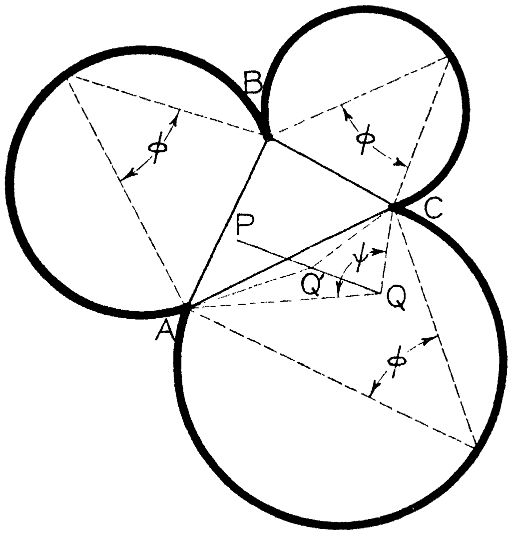

FIG. (8, 3)

This fact is obvious when Q is also a point of K. We need therefore only consider the case that $Q : \zeta$ is not a point of K. Then the angle ψ subtended at Q by K satisfies the inequality $\phi \leqq \psi < \pi$ and two points α and β can be found in K so that $\psi = \arg (\beta - \zeta)/(\alpha - \zeta)$. Let us choose any point $Q' : \zeta'$ lying on the segment PQ and let us set $\psi' = \arg (\beta - \zeta')/(\alpha - \zeta')$. Obviously, $\psi' > \psi$. Since the angle subtended at Q' by K cannot be less than ψ', we infer that it is greater than ϕ and that therefore Q' lies in the region $S(K, \phi)$.

In view of this discussion we may restate Th. (8,1) in the following form, which in the case $\gamma = 0$ is due to Nagy [2] but in the general case is due to Marden [4].

THEOREM (8,2). *If all the zeros and poles of each rational function $f_j(z)$ entering in eq.* (8,1) *lie in a closed convex region K and if the m_j ($j = 1, 2, \cdots, n$) are constants satisfying ineq.* (8,2), *then all the zeros of the linear combination $F(z) = \sum_{j=1}^{n} m_j f_j(z)$ lie in $S(K, \phi)$, a region which is star-shaped with respect to K and which consists of all points from which K subtends an angle of at least $\phi = (\pi - \gamma)/(p + q)$.*

We may add that in Th. (8,2) the region $S(K, \phi)$ may not be replaced by a smaller region. (Cf. Marden [4].) For, if $P : s$ is any point in $S(K, \phi)$, two

points $Q_1 : t_1$ and $Q_2 : t_2$ may be found in K such that $\sphericalangle Q_1PQ_2 = \phi$. Let us denote by d_1 and d_2 the distances of Q_1 and Q_2 from P respectively and by ω the angle formed by the ray PQ_1 with the positive real axis. Also let us define

$$k_1 = [(s - t_1)/d_1]^{p+q} \quad \text{and} \quad k_2 = [(s - t_2)/d_2]^{p+q}e^{i\gamma}.$$

Then, since $|k_1| = |k_2| = 1$,

$$\arg k_1 = (p + q)\omega$$

and

$$\arg k_2 = (p + q)(\omega + \phi) + \gamma = \pi + (p + q)\omega,$$

the vectors k_1 and k_2 are equal and opposite and thus

$$k_1 + k_2 = 0.$$

This means that the function

$$G(z) = [d_2^q(z - t_1)^p/d_1^p(z - t_2)^q] + e^{\gamma i}[d_1^q(z - t_2)^p/d_2^p(z - t_1)^q]$$

has a zero at the point s. In other words, every point s of $S(K, \phi)$ is a zero of at least one function $F(z)$ of type (8,1).

Th. (8,1) is a generalization of the Lucas Theorem (6,1) as may be seen by setting $\gamma = 0$, $p = 0$ and $q = 1$. Like the Lucas Theorem, it has various physical interpretations.

If $\gamma \neq 0$, $p = 0$ and $q = 1$, the function $F(z)$ in (8,1) has the form (2,3) and thus Th. (8,1) describes the location of the equilibrium points in a field of force due to complex masses m_j acting according to the inverse distance law. An example of such a field is the one due to both the charges carried by long straight wires at right angles to the z-plane and to the electromagnetic field induced by the currents flowing through these wires. Another example is the velocity field in the two-dimensional flow due to a vortex-source obtained by placing a source and vortex at the same point.

If $\gamma \neq 0$, $p = 1$ and $q = 0$, the zero of $F(z)$ is the "centroid" of a system of complex masses and thus Th. (8,1) describes the location of this centroid in relation to these particles.

As another application of Th. (8,2), let us introduce a polynomial $f(z)$ of degree p and n polynomials $h_j(z)$, $j = 1, 2, \cdots, n$, each of degree at most $p - 1$. Then

$$F(z) = f(z) - \frac{m_1h_1(z) + m_2h_2(z) + \cdots + m_nh_n(z)}{m_1 + m_2 + \cdots + m_n}$$

$$= \sum_{j=1}^{n} m_j[f(z) - h_j(z)]/\sum_{j=1}^{n} m_j$$

is a polynomial of type (8,1) with $q = 0$ and with

$$f(z) - h_j(z) \equiv (z - a_{j1})(z - a_{j2}) \cdots (z - a_{jp}).$$

The a_{jp} are clearly the points at which $f(z) = h_j(z)$ and the zeros of $F(z)$ are the points where

$$f(z) = \sum_{j=1}^{n} m_j h_j(z) / \sum_{j=1}^{n} m_j .$$

In other words, we have established the following *Mean-Value Theorem* for polynomials.

THEOREM (8,3). *Let $f(z)$ be a pth degree polynomial, let each $h_j(z)$ $(j = 1, 2, \cdots, n)$ be a polynomial of degree at most $p - 1$ and let m_j be complex constants satisfying ineq.* (8,2). *If all the points z at which $f(z) = h_j(z)$ for at least one j $(j = 1, 2, \cdots, n)$ lie in a convex region K, all the points at which*

$$f(z) = \sum_{j=1}^{n} m_j h_j(z) / \sum_{j=1}^{n} m_j \tag{8,5}$$

lie in the star-shaped region $S(K, (\pi - \gamma)/p)$.

Th. (8,3) is due to Marden [4]. When $\gamma = 0$, it reduces to the results of Nagy [2] and when in addition $h_j(z) = \text{const.}$, it reduces to the results stated by Jentsch [1] and proved by Fekete [2].

EXERCISES. Prove the following.

1. If the points a_{jk} and b_{jk} lie in a convex region K, then in the region $S(K, (\pi - \gamma)/(p + q))$ lies at least one of the points $z_1, z_2, \cdots, z_n$ which satisfy the relation

$$\sum_{j=1}^{n} m_j \frac{(z_j - a_{j1})(z_j - a_{j2}) \cdots (z_j - a_{jp})}{(z_j - b_{j1})(z_j - b_{j2}) \cdots (z_j - b_{jq})} = 0.$$

Hint: Assume the contrary.

2. If all the points at which a given pth degree polynomial $f(z)$ assumes n given values $c_1, c_2, \cdots, c_n$ are enclosed in a convex region K, and if the m_j are numbers satisfying (8,2), then all the points at which $f(z)$ assumes the average value

$$c = \sum_{j=1}^{n} m_j c_j / \sum_{j=1}^{n} m_j$$

lie in the star-shaped region $S(K, (\pi - \gamma)/p)$ [Marden 7 and 8; for cases $\gamma = 0$, Fekete 2 to 6 and Nagy 4].

3. Let K be a convex region which contains all the poles b_j of

$$f(z) = (z - a_1)(z - a_2) \cdots (z - a_p)/(z - b_1)(z - b_2) \cdots (z - b_q)$$

as well as all the points where $f(z)$ assumes the values $c_1, c_2, \cdots, c_n$. Let the m_j be constants satisfying ineq. (8,2) and let $c = \sum_{j=1}^{n} m_j c_j / \sum_{j=1}^{n} m_j$. Then $f(z) \neq c$ outside the star-shaped region $S(K, (\pi - \gamma)/(p + q))$.

4. For the values $r \leqq t \leqq s$ of the real variable t, let the equations $z = a_j(t)$ and $z = b_j(t)$ represent Jordan curves which lie in a convex region K and let $z = m(t)$ represent a Jordan curve which lies in a sector with vertex at the origin and with an angular opening of $\gamma < \pi$.

Let furthermore

$$f(z,t) = \prod_{j=1}^{n} \{[z - a_j(t)]/[z - b_j(t)]\}$$

and $F(z) = \int_r^s m(t)f(z, t)\, dt$. Then, $F(z) \neq 0$ outside the star-shaped region $S(K, (\pi - \gamma)/(p + q))$ [Marden 4].

5. Let $f(z) = \prod_{k=1}^{n} (z - z_k)$ and $g(z) = f(z + A) - Bf(z - A)$ where A and B are arbitrary complex numbers with, however, $0 < b = \arg B < \pi$. Then no zero Z of $g(z)$ may lie outside of all the lens-shaped regions defined by the inequalities $b/n \leqq \arg (z - z_k + A)/(z - z_k - A) \leqq \pi$, $k = 1, 2, \cdots, n$, or may lie inside all these regions [Nagy 8].

6. Let $f(z)$ be a real polynomial of degree n having n distinct zeros c_j which consist of the p pairs of conjugate imaginary zeros c_j, $c_{p+j} = \bar{c}_j$ $(j = 1, 2, \cdots, p;\ 2p \leqq n)$ and the $n - 2p$ real zeros c_j $(j = 2p + 1, 2p + 2, \cdots, n)$. Let $f_1(z)$ be a real polynomial of degree $n - 1$ which relative to $f(z)$ has the partial fraction development

$$\frac{f_1(z)}{f(z)} = \sum_{j=1}^{p} \left(\frac{\gamma_j}{z - c_j} + \frac{\bar{\gamma}_j}{z - \bar{c}_j}\right) + \sum_{j=2p+1}^{n} \frac{\gamma_j}{z - c_j}, \tag{8,6}$$

where $\gamma_j = m_j e^{i\mu_j}$ with $m_j > 0$ and μ_j real for all j and $\mu_j = 0$ for $j > 2p$. Let it be assumed that $|\mu_j| < \pi/2$ for $j \leqq 2p$. Let $K(c_j, \mu_j)$ be the circle which passes through the conjugate imaginary pair c_j, $\bar{c}_j$ and which has its center on the real axis at the point k_j such that angle $\bar{c}_j, c_j, k_j$ is μ_j. Then (a) any interval containing all the real zeros of $f(z)$ and all the points k_j $(j = 1, 2, \cdots, p)$ also contains all the real zeros of $f_1(z)$; (b) between two successive real zeros of $f(z)$ lie an odd number of zeros of $f_1(z)$; (c) any interval of the real axis not containing any zero of $f(z)$ and any interior point of any circle $K(c_j, \mu_j)$ contains at most one zero of $f_1(z)$ [Marden 17].

7. In eq. (8,6), assume that $m_j > 0$ for $j > 2p$ but $m_j > 0$ or < 0 for $j \leqq 2p$. Then each non-real zero of $f_1(z)$ lies either in at least one circle $K(c_j, \mu_j)$ corresponding to $m_j > 0$ or outside at least one circle $K(c_j, \mu_j)$ corresponding to $m_j < 0$ [Marden 17].

8. In eq. (8,6) assume that all $m_j > 0$. Let I: $\alpha \leqq x \leqq \beta$ be an interval of the real axis such that neither α nor β is a zero of $f(z)$ or an interior point of any circle $K(c_j, \mu_j)$. Let N be the configuration comprised of I and all the circles $K(c_j, \mu_j)$ which intersect I. Then, if N contains ν zeros of $f(z)$, it contains at least $\nu - 1$ and at most $\nu + 1$ zeros of $f_1(z)$ [Marden 17].

9. Let $f_0(z), f_1(z), \cdots, f_q(z)$ be the set of real polynomials such that for $k = 0, 1, \cdots, q - 1$

$$\frac{f_{k+1}(z)}{f_k(z)} = \sum_{j=1}^{p_k} \left(\frac{\gamma_{jk}}{z - c_{jk}} + \frac{\bar{\gamma}_{jk}}{z - \bar{c}_{jk}} \right) + \sum_{j=2p_k+1}^{n-k} \frac{\gamma_{jk}}{z - c_{jk}}$$

where $|\arg \gamma_{jk}| \leqq \omega_k < \pi/2$ for $j = 1, 2, \cdots, n - k$ and γ_{jk} and c_{jk} are real for $j > 2p_k$. For convenience, take $f_0(z) = f(z)$ and $c_{j0} = c_j$ for all j and set $\lambda_k = \cot [(\pi/4) - (\omega_k/2)]$ for all k. Let E_{jq} be the ellipse with center at the point $(c_j + \bar{c}_j)/2$, with a major axis $M_q \mid c_j - \bar{c}_j \mid$ along the axis of reals and with a minor axis $N_q \mid c_j - \bar{c}_j \mid$ where $N_q = \lambda_0\lambda_1 \cdots \lambda_{q-1}$ and $M_q^2 = \sum_{k=1}^{q} N_q^2$. Then each non-real zero of $f_q(z)$ lies in at least one ellipse E_{jq} $(j = 1, 2, \cdots, p)$ [Marden 17].

10. Let $\phi(z, c) = [1/(z - c)] + (1/c) + (z/c^2) + \cdots + (z^{k-1}/c^k)$ and let $F(z)$ be the real meromorphic function

$$F(z) = \sum_{j=1}^{\infty} A_j\phi(z,a_j) + \sum_{j=1}^{\infty} [B_j\phi(z,b_j) + \bar{B}_j\phi(z,\bar{b}_j)]$$

where A_j and a_j are real with $(A_j/a_j^k) > 0$ for all j, where $|\mu_j| < \pi/2$ for $\mu_j \equiv \arg (B_j/b_j^k) \pmod{2\pi}$ and where the series $\sum_{j=1}^{\infty} |A_j/a_j^{k+1}|$ and $\sum_{j=1}^{\infty} |B_j/b_j^{k+1}|$ are convergent. Then each non-real zero of $F(z)$ lies in at least one of the circles $K(b_j, \mu_j)$, $j = 1, 2, \cdots$ [Marden 17].

11. Let K be the smallest convex region enclosing all the zeros of $f(z)$, **a** polynomial of degree n. Then all the zeros of the mth derivative of $F(z) = 1/f(z)$ lie in the star-shaped region $S = S(K, \pi/m)$. Hint: Let $f_1(z) = f(\omega z + t) = c \prod_1^n (1 - z_k z)$. If t is any point outside S, ω may be chosen so that $0 < \arg z_k < \pi/m$ for all k. But $F(\omega z + t) = \sum_0^\infty c_k z^k$ with $c_m = F^{(m)}(t)/t! = c^{-1} \sum z_1^{k_1} z_2^{k_2} \cdots z_n^{k_n}$ and $k_1 + k_2 + \cdots + k_n = m$. By Th. (1,1), $F^{(m)}(t) \neq 0$ [Obrechkoff 8].

9. **Polynomial solutions of Lamé's differential equation.** In the previous section we studied the generalization of the Lucas Theorem from rational functions $F(z) = g(z)/f(z)$ whose decomposition into partial fractions has the form $\sum m_j(z - z_j)^{-1}$ involving real m_j to those whose decomposition has the form $\sum m_j g_j(z)/f_j(z)$ involving complex m_j. In this section we shall extend the Lucas Theorem to systems of partial fraction sums. We shall be principally interested in the systems which arise in the study of the polynomial solutions of the generalized Lamé differential equation

$$\text{(9,1)} \qquad \frac{d^2w}{dz^2} + \left(\sum_{j=1}^{p} \frac{\alpha_j}{z - a_j} \right) \frac{dw}{dz} + \frac{\Phi(z)}{\prod_{j=1}^{p} (z - a_j)} w = 0.$$

By a straightforward application of the method of undetermined coefficients Heine [1] demonstrated the existence of $C(n + p - 2, p - 2)$ polynomials $V(z)$ of degree $p - 1$ such that for $\Phi(z) = V(z)$ eq. (9,1) has a polynomial solution $S(z)$ of degree n. We shall call each $V(z)$ a *Van Vleck polynomial* and the corresponding $S(z)$ a *Stieltjes polynomial* in recognition of the fact that

Van Vleck [1] and Stieltjes [2] were the first to study the distribution of the zeros of the polynomials $V(z)$ and $S(z)$ respectively. (Cf. ex. (9,1) and ex. (9,2).)

If $S(z) = (z - z_1)(z - z_2) \cdots (z - z_n)$ is a Stieltjes polynomial, it follows from (9,1) that

$$S''(z_k) + \left(\sum_{j=1}^{p} \alpha_j/(z_k - a_j) \right) S'(z_k) = 0 \qquad (k = 1, 2, \cdots, n). \tag{9,2}$$

If $S'(z_k) = 0$ but $S''(z_k) \neq 0$, eq. (9,2) would be satisfied only if $z_k = a_j$ for some value of j. If $S'(z_k) = S''(z_k) = 0$, the differential equations obtained on successively differentiating (9,1) would show that all derivatives of $S(z)$ would vanish at $z = z_k$, an impossibility since $S_n(z)$ is an nth degree polynomial. If $S'(z_k) \neq 0$, we may write

$$S(z) = (z - z_k)T(z), \qquad T(z_k) \neq 0,$$

and obtain

$$\frac{S''(z_k)}{S'(z_k)} = \frac{2T'(z_k)}{T(z_k)} = \sum_{j=1, j \neq k}^{n} \frac{2}{z_k - z_j}.$$

Consequently, every zero z_k of $S(z)$ is either a point a_j or a solution of the system

$$\sum_{j=1}^{p} \frac{\alpha_j}{z_k - a_j} + \sum_{j=1, j \neq k}^{n} \frac{2}{z_k - z_j} = 0, \qquad k = 1, 2, \cdots, n. \tag{9,3}$$

In the latter case, the zero z_k has an interpretation similar to that assigned to the zeros of (6,2). The term $\alpha_j(\bar{a}_j - \bar{z}_k)^{-1}$ in the conjugate imaginary of (9,3) may be regarded as the force upon a unit mass at the variable point z_k due to the mass α_j situated at the fixed point a_j. The term $(\bar{z}_j - \bar{z}_k)^{-1}$ may be regarded as the force upon the unit mass at z_k due to the unit mass at the variable point z_j. In other words, the system (9,3) defines the z_k as the points of equilibrium of n movable unit particles in a field due to p fixed particles a_k of mass $\alpha_k/2$.

Likewise, if t_k is a zero of the Van Vleck polynomial $V(z)$ corresponding to $S(z)$, then

$$S''(t_k) + \left[\sum_{j=1}^{p} \alpha_j/(t_k - a_j) \right] S'(t_k) = 0. \tag{9,4}$$

Thus t_k is either a zero of $S'(z)$, which we may write as

$$S'(z) = n(z - z_1')(z - z_2') \cdots (z - z_{n-1}'),$$

or $S'(t_k) \neq 0$ and

$$\left[\sum_{j=1}^{p} \alpha_j/(t_k - a_j) \right] + \left[\sum_{j=1}^{n-1} 1/(t_k - z_j') \right] = 0. \tag{9,5}$$

We leave to the reader the physical interpretation of the t_k.

The location of the zeros of $S(z)$ and of $V(z)$ has been studied by Stieltjes, Van Vleck, Bôcher and Pólya in the case $\gamma = 0$, their results being given below in exs. (9,1), (9,2) and (9,3). For the general case we shall now prove a theorem due to Marden [5].

THEOREM (9,1). *If*

$$|\arg \alpha_j| \leqq \gamma < \pi/2, \qquad j = 1, 2, \cdots, p, \tag{9,6}$$

and if all the points a_j lie in a circle C of radius r, then the zeros of every Stieltjes polynomial and the zeros of every Van Vleck polynomial lie in the concentric circle C' of radius $r' = r \sec \gamma$.

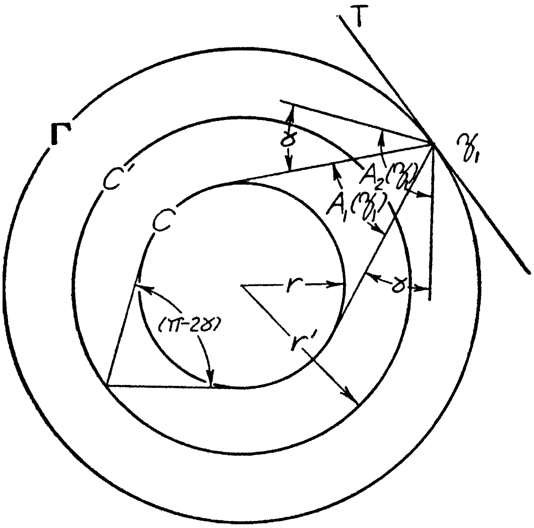

FIG. (9, 1)

To prove the first part of this theorem, let us suppose that the Stieltjes polynomial

$$S(z) = (z - z_1)(z - z_2) \cdots (z - z_n)$$

has some zeros outside C' and that among these the one farthest from the center of C is z_1. (See fig. (9,1).) Then at z_1 circle C would subtend an angle $A_1(z_1)$ of magnitude less than $\pi - 2\gamma$. Through z_1 let us draw the circle Γ concentric with C and let us draw the line T tangent to Γ at z_1. By the assumption concerning z_1, all the points z_j lie in or on the circle Γ and hence the quantities $(\bar{z}_j - \bar{z}_1)^{-1}$ are represented by vectors drawn from z_1 to points on the side of T containing circle C. Furthermore, since the quantity $(\bar{a}_j - \bar{z}_1)^{-1}$ may be represented by a vector drawn from z_1 and lying in the angle $A_1(z_1)$, the quantity $\alpha_j(\bar{a}_j - \bar{z}_1)^{-1}$ may, due to (9,6), be represented by a vector drawn from z_1 and lying in the angle $A_2(z_1)$ formed by adding an angle γ to both sides of $A_1(z_1)$.

The angle $A_2(z_1)$, being in magnitude less than $2\gamma + (\pi - 2\gamma) = \pi$, lies on the same side of T as does C. In short, both types of terms $(\bar{z}_j - \bar{z}_1)^{-1}$ and $\alpha_j(\bar{a}_j - \bar{z}_1)^{-1}$ entering in eq. (9,3) are representable by vectors drawn from z_1 to points on the same side of T. This means according to Th. (1,1) that the left-side of eq. (9,3) cannot vanish. Since this result contradicts eq. (9,3), our conclusion is that the point z_1 and consequently all z_j must lie in C'.

With the first part of Th. (9,1) thus proved, it remains to consider the second part that concerns the zeros of $V(z)$, the Van Vleck polynomial corresponding to $S(z)$. Since we now know that all the zeros z_j of $S(z)$ lie in circle C'', we may

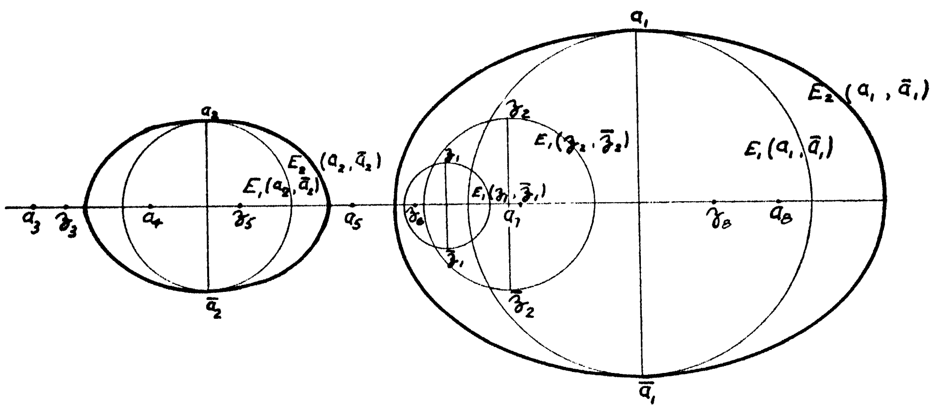

FIG. (9, 2)

infer from Theorem (6,2) that all the zeros z_j' of the derivative $S'(z)$ also lie in C'. Let us assume concerning $V(z)$ that its zero t_1, farthest from the center of C, were outside C' and let us draw through t_1 a circle Γ and its tangent T. By then repeating essentially the same reasoning as in the first part, we can show that our assumption concerning t_1 implies the non-vanishing of the left-side of eq. (9,5) in contradiction to the hypothesis of the theorem.

In the case of real, positive α_j, the part of Th. (9,1) that concerns Stieltjes polynomials may be regarded as a generalization of the Lucas Theorem (Th. 6,2). For this same case, Walsh [8] has given the following generalization of the Jensen Theorem (Th. 7,1). (See fig. 9,2.)

THEOREM (9,2). *Let the α_j in eq. (9,1) be positive real numbers and let the non-real a_j occur in conjugate imaginary pairs. Let $E_m(a, \bar{a})$ denote the ellipse whose minor axis is the line-segment joining points a and $\bar{a}$ and whose major axis is $m^{1/2}$ times as long as the minor axis. Then no non-real zero of any Stieltjes polynomial having m pairs of non-real zeros may lie exterior to all the ellipses $E_m(a_j, \bar{a}_j)$, $j = 1, 2, \cdots, p$.*

It is to be noted that $E_1(a, \bar{a})$ is the Jensen circle of the pair $(a, \bar{a})$.

In the proof of this theorem, we shall use two lemmas. The first is one which may be easily verified by elementary calculus; namely,

LEMMA (9,2a). *The circles whose diameters are the vertical chords of the ellipse $E_{m-1}(a, \bar{a})$ lie in the closed interior of the ellipse $E_m(a, \bar{a})$ and have this ellipse as their envelope.*

For the statement of the second lemma, let us write $S(z)$ in the form

$$(9,7)\quad S(z) = (z - z_1)(z - \bar{z}_1) \cdots (z - z_m)(z - \bar{z}_m)(z - z_{2m+1}) \cdots (z - z_n)$$

with the z_j , $j > 2m$, representing the real zeros of $S(z)$. The second lemma is then the following.

LEMMA (9,2b). *If the non-real zero z_1 of $S(z)$ lies outside the Jensen circles $E_1(a_j , \bar{a}_j)$, $j = 1, 2, \cdots , p$, it lies inside at least one Jensen circle $E_1(z_j , \bar{z}_j)$, $2 \leqq j \leqq m$.*

For, eq. (9,3) becomes for (9,7) and for $k = 1$,

$$(9,8)\quad \sum_{j=1}^{p} \frac{\alpha_j}{z_1 - a_j} + \frac{2}{z_1 - \bar{z}_1} + \sum_{j=2}^{m} \left(\frac{2}{z_1 - z_j} + \frac{2}{z_1 - \bar{z}_j}\right) + \sum_{j=2m+1}^{n} \frac{2}{z_1 - z_j} = 0.$$

Except for the term $(z_1 - \bar{z}_1)^{-1}$, eq. (9,8) has the form of eq. (6,2). If z_1 were also outside the Jensen circles of the points z_j , $2 \leqq j \leqq m$, then we could apply the reasoning used to prove Th. (7,1). Thus for all terms in (9,8), except possibly $(z_1 - \bar{z}_1)^{-1}$, the sign of the imaginary part would be that of $\mathrm{sg}(-y_1)$. But, since $(z_1 - \bar{z}_1)^{-1} = -2i/y_1$, the sign of imaginary part of all terms would be that of $\mathrm{sg}(-y_1)$. That is, if z_1 were outside of the Jensen circles for all the a_j , $1 \leqq j \leqq p$, and all the z_j , $2 \leqq j \leqq m$, then it would not satisfy eq. (9,8).

Now, to prove Th. (9,2), let us assume that point z_1 is exterior to all the Jensen circles $E_1(a_j , \bar{a}_j)$. By Lemma (9,2b) point z_1 is interior to, say, $E_1(z_2 , \bar{z}_2)$. If, then, z_2 is also exterior to all the Jensen circles $E_1(a_j , \bar{a}_j)$, it lies interior to, say, $E_1(z_3 , \bar{z}_3)$, and so forth. Eventually, we must come to a value of k, $k \leqq m$, such that, although the point z_{k-1} lies exterior to all the circles $E_1(a_j , \bar{a}_j)$ and thus lies interior to the circle $E_1(z_k , \bar{z}_k)$, the point z_k lies interior to at least one circle $E_1(a_j , \bar{a}_j)$, say $E_1(a_1 , \bar{a}_1)$.

Now applying Lem. (9,2a), we see that circle $E_1(z_k , \bar{z}_k)$ lies in ellipse $E_2(a_1 , \bar{a}_1)$; that circle $E_1(z_{k-1} , \bar{z}_{k-1})$ therefore lies in ellipse $E_3(a_1 , \bar{a}_1)$, etc., finally, that circle $E_1(z_2 , \bar{z}_2)$ lies in the ellipse $E_k(a_1 , \bar{a}_1)$. Since however, $k \leqq m$, ellipse $E_k(a_1 , \bar{a}_1)$ lies in the ellipse $E_m(a_1 , \bar{a}_1)$. Thus we have completed the proof of Th. (9,2).

Instead of assuming that the α_j are positive real numbers, let us suppose that the α_j corresponding to a pair a_j , $\bar{a}_j$ form a conjugate imaginary pair. We may then prove the following two theorems.

THEOREM (9,3). *If the a_j and the corresponding α_j are real or appear in conjugate imaginary pairs and if $|\arg \alpha_j| < \pi/2$ for all j, then the zeros of every*

Stieltjes polynomial and those of the corresponding Van Vleck polynomial lie in the smallest convex region which encloses both all the real points a_i and all the ellipses having the pairs of points a_i and $\bar{a}_i$ as foci and having eccentricities equal to cos (arg α_i).

THEOREM (9,4). *Under the hypotheses of Th.* (9,3), *let $S(z)$ be a Stieltjes polynomial possessing k pairs of conjugate imaginary zeros and let $V(z)$ be the corresponding Van Vleck polynomial. Corresponding to each conjugate imaginary pair a_i, $\bar{a}_i$ let the real point e_i be located such that angle $\bar{a}_i$, a_i, e_i is* arg α_i *and let $E(a_i, q)$ denote the ellipse with center at c_i, with a minor axis $m_i = 2 \mid c_i - e_i \mid$ parallel to the imaginary axis and with a major axis $q^{1/2}m_i$. Then every non-real zero of $S(z)$ lies in at least one of the ellipses $E(a_i, k)$ and every non-real zero of $V(z)$ lies in at least one of the ellipses $E(a_i, k + 2)$.*

Theorem (9,3) is a Lucas type of theorem which may be proved with the aid of the lemma stated in ex. (9,5). The part which concerns the Stieltjes polynomials was first proved in Vuille [1]. The theorem in its entirety was established in Marden [5].

Th. (9,4) is a Jensen type of theorem which is a generalization of Th. (9,2) and which may be established with the aid of ex. (8,6) and of the method of proof used for Th. (9,2). Th. (9,4) is due to Marden [20].

EXERCISES. Prove the following.

1. If in eq. (9,6) $\gamma = 0$ and if all the a_i lie on a segment σ of the real axis, the zeros of every Stieltjes polynomial will also lie on σ [Stieltjes 2].
2. Under the hypothesis of ex. 1, the zeros of every Van Vleck polynomial will also lie on σ [Van Vleck 1].
3. If $\gamma = 0$, any convex region K containing all the points a_i will also contain all the zeros of every Stieltjes polynomial [Bôcher 1, Klein 1, and Pólya 1].
4. Under the hypothesis of ex. 3, K also contains all the zeros of every Van Vleck polynomial [Marden 5].
5. Let the "mass" α be at point $z = \lambda i$ $(\lambda > 0)$ and the "mass" $\bar{\alpha}$ at point $z = -\lambda i$. The resultant force

$$\bar{\alpha}(-\lambda i - \bar{z}_1)^{-1} + \alpha(\lambda i - \bar{z}_1)^{-1}$$

at z_1 due to these two masses has a line of action which intersects the ellipse with $\pm\lambda i$ as foci and with cos (arg α_i) as eccentricity [Marden 5].

6. The zeros of the Legendre polynomials $P_n(z)$ lie on the interval $-1 \leqq z \leqq 1$ of the real axis. Hint: The Legendre polynomials $P_n(z)$ may be defined as the solutions of the differential equation

$$(1 - z^2)P_n''(z) - 2zP_n'(z) + n(n + 1)P_n(z) = 0.$$

7. If the differential equation $w'' + A(z)w' + B(z)w = 0$, where $A(z)$ and $B(z)$ are functions analytic in a region R, has as a solution an nth degree polynomial $P(z)$, the zeros of $P(z)$ in R are the points of equilibrium of n movable

unit particles in the plane field of force whose magnitude and direction at any point z of R is that of the vector $\overline{A(\bar{z})}$. The movable particles attract one another according to the inverse distance law [Bôcher 4].

8. The zeros of the Hermite polynomials $H_n(z)$ are all real and distinct. Hint: By definition, $w = H_n(z)$ is a solution of the differential equation $w'' - xw' + nw = 0$. Use ex. (9,7) [Bôcher 4].

CHAPTER III

INVARIANTIVE FORMULATION

10. The derivative under linear transformations. In the last two chapters we were interested in proving some theorems concerning the zeros of the logarithmic derivative of the function

$$(10,1) \qquad f(z) = \prod_{j=1}^{p} (z - z_j)^{m_j}, \qquad n = \sum_{j=1}^{p} m_j ,$$

and in extending these theorems to more general rational functions and to certain systems of rational functions. We obtained these results largely by use of Th. (1,1) and Th. (1,4).

We now wish to see what further generalizations, if any, may be derived by use of the method of conformal mapping. For instance, we know by virtue of the Lucas Th. (6,2) that any circle C containing all the zeros of a polynomial $f(z)$ also contains all the zeros of the derivative $f'(z)$ of $f(z)$. Since we may map the closed interior of C conformally upon the closed exterior of a circle C', can we then infer that, if all the zeros of $f(z)$ lie exterior to C', so do all the zeros of $f'(z)$? Certainly not in general, as we see from the example $f(z) = z^3 - 8$ with C' taken as the exterior of the circle $|z| = 1$.

Let us consider how to generalize Th. (6,2) so as to obtain a result which will be invariant relative to the nonsingular linear transformations

$$(10,2) \qquad z = \frac{\alpha Z + \beta}{\gamma Z + \delta}, \qquad \Delta = \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} \neq 0.$$

Specifically, let us denote by Z_j the points into which the zeros z_j of $f(z)$ are transformed by (10,2) and by Z_k' the points into which the zeros z_k' of $f'(z)/f(z)$ are transformed; that is

$$(10,3) \qquad z_j = (\alpha Z_j + \beta)/(\gamma Z_j + \delta), \qquad z_k' = (\alpha Z_k' + \beta)/(\gamma Z_k' + \delta).$$

Clearly, the Z_j are the zeros of $F(Z)$, the transform of $f(z)$, namely,

$$(10,4) \qquad F(Z) = (\gamma Z + \delta)^n f\left(\frac{\alpha Z + \beta}{\gamma Z + \delta}\right).$$

The Z_k', however, are not in general the zeros of the logarithmic derivative of $F(Z)$. Let us inquire as to the choice of the m_j necessary and sufficient for a finite Z_k' to be such a zero.

The logarithmic derivative of $F(Z)$ calculated from eq. (10,4) is

$$(10,5) \qquad \frac{F'(Z)}{F(Z)} = \frac{\gamma n}{\gamma Z + \delta} + \left\{ f'\left(\frac{\alpha Z + \beta}{\gamma Z + \delta}\right) \frac{\Delta}{(\gamma Z + \delta)^2} \left[f\left(\frac{\alpha Z + \beta}{\gamma Z + \delta}\right)\right]^{-1} \right\}.$$

We thereby obtain

$$\frac{F'(Z_k')}{F(Z_k')} = \frac{\gamma n}{\gamma Z_k' + \delta}. \tag{10,6}$$

Thus a necessary and sufficient condition for $F'(Z_k') = 0$ if $Z_k' \neq \infty$ is that $\gamma n = 0$.

This condition will be satisfied if we choose $\gamma = 0$; that is, if we select for (10,2) any nonsingular linear integral transformation

$$z = AZ + B, \qquad A \neq 0. \tag{10,7}$$

Thus, if we restrict the transformations to translations, rotations and those of similitude, the zeros of $f'(z)/f(z)$ transform into those of $F'(Z)/F(Z)$ when the m_j are chosen as arbitrary positive or negative numbers.

To satisfy the condition when $\gamma \neq 0$ and $Z_k' \neq \infty$, we must choose $n = 0$. This implies that not all m_j may be of the same sign. In particular, it excludes the case that all the m_j are positive integers. In other words, under the general transformation (10,2) the zeros of the logarithmic derivative of a polynomial are not carried into the zeros of the logarithmic derivative of $F(Z)$.

Eq. (10,6) does, however, suggest that, in place of the derivative $f'(z)$ of a given nth degree polynomial $f(z)$, there be introduced the function

$$f_1(z) = nf(z) - (z - z_0)f'(z). \tag{10,8}$$

Like $f'(z)$, $f_1(z)$ is a polynomial of degree $n - 1$. It is furthermore a generalization of the derivative in the sense that

$$\lim_{z_0 \to \infty} [f_1(z)/(z_0 - z)] = f'(z). \tag{10,9}$$

The function $f_1(z)$ has been called by Laguerre [1, p. 48] the "émanant" of $f(z)$ and by Pólya-Szegö [1, vol. 2, p. 61] "the derivative of $f(z)$ with respect to the point z_0," but we shall call $f_1(z)$ the polar derivative of $f(z)$ with respect to the pole z_0 or simply the *polar derivative* of $f(z)$.

The zeros of the polar derivative are:
(a) the point z_0 if $f(z_0) = 0$;
(b) the multiple zeros of $f(z)$, and
(c) the zeros of the function

$$\frac{f_1(z)}{(z - z_0)f(z)} = \frac{n}{z - z_0} - \sum_{j=1}^{p} \frac{m_j}{z - z_j}. \tag{10,10}$$

Since (10,10) is the logarithmic derivative of

$$-f(z)(z - z_0)^{-n},$$

a function of type (10,1) with a total "degree" of zero, the zeros of (10,10) and hence those of $f_1(z)$ are invariant under the general linear transformation (10,2).

In order to associate the polar derivative $f_1(z)$ with a more familiar invariant, let us introduce the homogeneous co-ordinates (ξ, η) by substituting $z = \xi/\eta$ into $f(z)$ and $f_1(z)$. Thus,

$$F(\xi, \eta) = \eta^n f(\xi/\eta),$$

$$F_1(\xi, \eta) = \eta^{n-1}\eta_0 f_1(\xi/\eta)$$

$$= \frac{\eta_0}{\eta}\left\{[nF(\xi, \eta)] - \frac{(\xi\eta_0 - \eta\xi_0)}{\eta_0}\frac{\partial}{\partial\xi}F(\xi, \eta)\right\}.$$

Since, as a homogeneous function of degree n, $F(\xi, \eta)$ satisfies the Euler identity

$$nF(\xi, \eta) = \xi\frac{\partial F}{\partial\xi} + \eta\frac{\partial F}{\partial\eta},$$

we find

$$F_1(\xi, \eta) = \xi_0\frac{\partial F}{\partial\xi} + \eta_0\frac{\partial F}{\partial\eta}.$$

In short, upon the introduction of homogeneous co-ordinates, the polynomial $f(z)$ transforms into a homogeneous function $F(\xi, \eta)$ and $f_1(z)$ into $F_1(\xi, \eta)$, the first polar of $F(\xi, \eta)$. This result provides further evidence of the invariant character of the polar derivative.

Exercises. Prove the following.

1. If the zeros of a polynomial $f(z)$ are symmetric in a line L, then between two successive zeros of $f(z)$ on L lie an odd number of zeros of its derivative $f'(z)$ and any interval of L which contains all the zeros of $f(z)$ lying on L also contains all the zeros of $f'(z)$ lying on L. Hint: Apply (10,7) to Rolle's Theorem.

2. Let $z = g(Z)$ be a rational function which has as its only poles those of multiplicities q_j at the points Q_j with $j = 1, 2, \cdots, k$. Let furthermore $h(Z) = \prod_{j=1}^{k}(Z - Q_j)^{q_j}$ and $F(Z) = h(Z)^n f(g(Z))$, where $f(z)$ is the function (10,1). Then a given zero z_j' of $f'(z)/f(z)$ is transformed by $z = g(Z)$ into a zero Z_j' of $F'(Z)/F(Z)$ if $h'(Z_j') = 0$ whereas all zeros z_j' are transformed into zeros Z_j' if $n = 0$.

11. Covariant force fields. In order to throw some further light upon the invariant character of the zeros, not merely of the polar derivative of a polynomial, but also of the logarithmic derivative of any function $f(z)$ of type (10,1) with $n = 0$, we shall use a physical interpretation similar to that in sec. 3 coupled with the method of stereographic projection. (See Fig. 11,1).

At the fixed points P_j of a unit sphere S let us place masses m_j which repel (attract if $m_j < 0$) a unit mass at the variable point P of S according to the inverse distance law. Let us denote by $\Phi(P)$ resultant force at P.

By drawing lines from the north pole N of S through the points P and P_j let

us project P and P_i stereographically upon the equatorial plane of S into the points z and z_i, respectively. At the points z_i let us place masses m_i which repel (attract if $m_i < 0$) a unit mass at z according to the inverse distance law. Let us denote by $\phi(z)$ the resultant force at z.

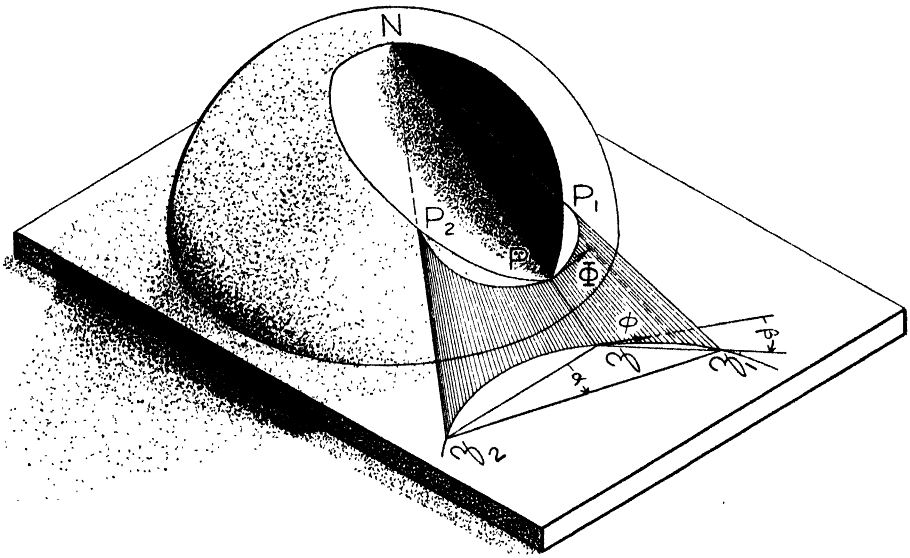

FIG. (11, 1)

We ask: what is the relation between the resultant force $\Phi(P)$ in the spherical field and the resultant force $\phi(z)$ in the corresponding plane field?

The answer to the question, given by Bôcher [4], is contained in

THEOREM (11,1). *Let $\Phi(P)$ be the resultant force upon a unit mass at a point P of a unit sphere S due to masses m_i at the p points P_i of S. Let z and z_i be the points into which P and P_i are carried by stereographic projection upon the equatorial plane of S. Let $\phi(z)$ be the resultant force upon a unit mass at z due to masses m_i at the points z_i. If the total mass $n = m_1 + m_2 + \cdots + m_p = 0$, then the force $\Phi(P)$ may be represented by a vector which is tangent to S and which projects into the vector* $[(1 + |z|^2)/2]\phi(z)$.

To establish this theorem, we shall need

LEMMA (11,1). *The lines of force in a field due to a mass $-m$ at point Q_1 and a mass $+m$ at point Q_2 are circles through Q_1 and Q_2. The resultant force $\phi(Q)$ upon a unit mass at any third point Q has a magnitude $m(Q_1Q_2)/(QQ_1)(QQ_2)$ and is directed along circle Q_1QQ_2 towards the negative mass.*

To prove this lemma, let us introduce complex numbers in the plane determined by the three points Q, Q_1 and Q_2 and denote their co-ordinates by z, z_1 and z_2 respectively. According to sec. 3,

$$(11,1) \qquad \phi(Q) = \frac{m}{\bar{z} - \bar{z}_2} - \frac{m}{\bar{z} - \bar{z}_1} = \frac{m(\bar{z}_2 - \bar{z}_1)}{(\bar{z} - \bar{z}_2)(\bar{z} - \bar{z}_1)}.$$

Obviously, $\phi(Q)$ has the required magnitude. As to its direction,

$$\arg \phi(Q) = \arg m - \arg (z_2 - z_1) + \arg (z - z_2) + \arg (z - z_1),$$

whence (see fig. 11,1)

$$\arg (z_1 - z) - \arg \phi(Q) = \arg (z_1 - z_2) - \arg (z - z_2) - \arg m.$$

That is,

$$\beta = \alpha - \arg m.$$

Thus $\beta = \alpha$ if $m > 0$, but $\beta = \alpha + \pi$ if $m < 0$, so that $\phi(Q)$ has also the required direction.

We proceed now to the proof of Th. (11,1).

Let us place at the north pole N of S p additional masses $(-m_j)$. Since by hypothesis their total mass $(-n) = 0$, the resultant force due to the augmented system consisting of these new masses and of the original masses m_j at P_j is the same as for the original system. The augmented system may, however, be considered as comprised of the p pairs of masses, m_j at P_j and $-m_j$ at N. According to Lemma (11,1) the jth pair acts upon a unit mass at P with a force $\Phi_j(P)$ tangent to the circle C_j through the points P, P_j and N. Since for every j the circle C_j lies on the sphere S, the resultant force $\Phi(P)$ due to all p pairs is tangent to the sphere S. Furthermore, since point N projects into the point $z = \infty$, the circle C_j projects into the straight line through z and z_j and vector $\Phi_j(P)$ projects into the vector directed either from z_j to z or from z to z_j according as $m_j > 0$ or $m_j < 0$.

To compare the magnitudes of these vectors, let us recall the relation between the co-ordinates of $P : (\xi, \eta, \zeta)$ and those of its projection $z = x + iy$; namely,

$$(11,2) \qquad x = \frac{\xi}{1 - \zeta}, \qquad y = \frac{\eta}{1 - \zeta}, \qquad x^2 + y^2 + 1 = \frac{2}{(1 - \zeta)}.$$

Hence, for the square of the magnitude of the force $\phi_j(z)$ due to the mass m_j at z_j, we have

$$| \phi_j(z) |^2 = \frac{m_j^2}{(x - x_j)^2 + (y - y_j)^2}$$

$$= \frac{m_j^2(1 - \zeta_j)^2(1 - \zeta)^2}{[\xi(1 - \zeta_j) - \xi_j(1 - \zeta)]^2 + [\eta(1 - \zeta_j) - \eta_j(1 - \zeta)]^2}.$$

On squaring out the denominator and on using the fact that, being on the sphere S, the points (ξ, η, ζ) and (ξ_j, η_j, ζ_j) satisfy the equations

$$(11,3) \qquad \xi^2 + \eta^2 = 1 - \zeta^2, \qquad \xi_j^2 + \eta_j^2 = 1 - \zeta_j^2,$$

we obtain

$$(11,4) \qquad | \phi_i(z) |^2 = \frac{m_i^2(1 - \zeta)(1 - \zeta_i)}{2(1 - \xi\xi_i - \eta\eta_i - \zeta\zeta_i)}.$$

Similarly on using (11,2), we obtain

$$| \Phi_i(P) |^2 = \frac{m_i^2(NP_i)^2}{(PP_i)^2(PN)^2}$$

$$= \frac{m_i^2[\xi_i^2 + \eta_i^2 + (\zeta_i - 1)^2]}{[(\xi - \xi_i)^2 + (\eta - \eta_i)^2 + (\zeta - \zeta_i)^2][\xi^2 + \eta^2 + (\zeta - 1)^2]}.$$

Consequently,

$$(11,5) \qquad | \Phi_i(P) |^2 = \frac{m_i^2(1 - \zeta_i)}{2[1 - \xi\xi_i - \eta\eta_i - \zeta\zeta_i][1 - \zeta]}.$$

From (11,2), (11,4) and (11,5) it then follows that

$$\left| \frac{\Phi_i(P)}{\phi_i(z)} \right|^2 = \frac{1}{(1 - \zeta)^2} = \frac{(1 + x^2 + y^2)^2}{4}.$$

By applying this result to each pair N, P_i, we bring to completion our proof of Th. (11,1).

From this theorem, we derive the important

COROLLARY (11,1). *The points of equilibrium in the spherical force field project into the points of equilibrium in the corresponding plane force field.*

For obviously, $\Phi(P) = 0$ if and only if $\phi(z) = 0$.

12. **Circular regions.** In the preceding two sections we were able to associate with every nth degree polynomial $f(z)$ an $(n - 1)$st degree polynomial called the polar derivative of $f(z)$, namely

$$f_1(z) = nf(z) + (\zeta - z)f'(z),$$

whose zeros remain invariant under the linear transformations (10,2). Since $f_1(z)$ is a generalization of the ordinary derivative, its zeros may be expected to satisfy some invariant form of the Lucas Theorem (6,2) that any circle C containing all the zeros of $f(z)$ also contains all the zeros of $f'(z)$. In order to find the corresponding theorem for the polar derivative, we need to consider the class of regions which includes the interior of a circle as a special case and which remains invariant under the transformation (10,2). As is well known, this is the class of so-called *circular regions*, consisting of the closed interiors or exteriors of circles and the closed half-planes.

In our subsequent work involving circular regions we shall find the following lemma very useful.

LEMMA (12,1). *Let* $C(z) \equiv |z - \alpha|^2 - r^2$, *so that* $C(z) = 0$ *is the equation of the circle* C *with center at point* α *and radius* r. *Let* z_1, Z *and* w_1 *be any three points connected by the relation* $w_1 = (\bar{Z} - \bar{z}_1)^{-1}$ *and let* C' *be the circle with center at* α' *and radius* r', *where*

$$\alpha' = (Z - \alpha)/C(Z) \qquad \text{and} \qquad r' = r/|C(Z)|. \tag{12,1}$$

Then the point w_1 *lies inside or outside the circle* C' *according as the circle* C *does or does not separate the two points* Z *and* z_1.

To prove this lemma, let us calculate $C'(w_1) = |w_1 - \alpha'|^2 - r'^2$.

$$C'(w_1) = [(Z - \bar{z}_1)^{-1} - (Z - \alpha)C(Z)^{-1}][(Z - z_1)^{-1}$$
$$- (\bar{Z} - \bar{\alpha})C(Z)^{-1}] - [r^2/C(Z)^2]$$
$$= |Z - z_1|^{-2} + C(Z)^{-1} - [(Z - \alpha)(\bar{Z} - \bar{z}_1)$$
$$+ (\bar{Z} - \bar{\alpha})(Z - z_1)]\, |Z - z_1|^{-2}C(z)^{-1}.$$

Using now the identity $A\bar{B} + \bar{A}B = |A|^2 + |B|^2 - |A - B|^2$ in the form

$$(Z - \alpha)(\bar{Z} - \bar{z}_1) + (\bar{Z} - \bar{\alpha})(Z - z_1)$$
$$= |Z - \alpha|^2 + |Z - z_1|^2 - |z_1 - \alpha|^2, \tag{12,2}$$

we obtain $C'(w_1) = |w_1|^2[C(z_1)/C(Z)]$.

If one of the points z_1 and Z is inside and the other is outside C, then $C(z_1)/C(Z) < 0$ and hence $C'(w_1) < 0$, implying that w_1 is inside circle C'. If, however, the points z_1 and Z are both inside or both outside circle C, then $C(z_1)/C(Z) > 0$ and hence $C'(w_1) > 0$, implying that w_1 is outside circle C'. This completes the proof of Lemma (12,1).

EXERCISES. Using the above equations, prove the following.

1. If the circle C passes through the point Z, then C' is a straight line passing through Z.
2. If C is the straight line $C(z) = \bar{\alpha}z + \alpha\bar{z} + b = 0$, b real, then C' is a circle passing through Z.
3. If the circle C passes through the point z_1 but not through the point Z, then C' is a circle passing through the point w_1.

13. **Zeros of the polar derivative.**

We are now ready to state the invariant form of the Lucas Theorem (Th. 6,2) due to Laguerre [1].

LAGUERRE'S THEOREM. (Th. 13,1). *If all the zeros* z_j *of the nth degree polynomial* $f(z)$ *lie in a circular region* C *and if* Z *is any zero of*

$$f_1(z) = nf(z) + (\zeta - z)f'(z), \tag{13,1}$$

the polar derivative of $f(z)$, then not both points Z and ζ may lie outside of C. Furthermore, if $f(Z) \neq 0$, any circle K through Z and ζ either passes through all the zeros of $f(z)$ or separates these zeros.

Because of the importance of Laguerre's theorem to our subsequent investigations, we shall give two proofs of it and also suggest a third in ex. (13,1).

The first proof will use the results of section 11 concerning spherical force fields. Let us assume that Z and ζ are both exterior to the region C. Since all the zeros of $f(z)$ lie in C, it follows that $f(Z) \neq 0$ and, hence, also $Z \neq \zeta$. Through Z a circle Γ may be drawn which separates the region C from the point ζ. As a zero of $f_1(z)$, Z must satisfy the equation

$$f_1(Z)/[(\zeta - Z)f(Z)] = -[n/(Z - \zeta)] + [f'(Z)/f(Z)] = 0 \tag{13,2}$$

and consequently must be an equilibrium point in a plane force field due to particles of total mass zero. With this plane force field may be associated a spherical force field in which points P_j, P and Q and circles C' and Γ' correspond respectively to points z_j, Z and ζ and circles C and Γ and in which the mass at P_j is m_j and the mass at Q is $-n = -(m_1 + m_2 + \cdots + m_p)$. The force Φ_j at P due to the pair consisting of m_j at P_j and of $(-m_j)$ at Q acts in the direction of the circular arc P_jPQ and hence towards the side of circle Γ' not containing C'. The vectors Φ_j are consequently all drawn from P to points on the same side of the tangent line to Γ' at P. According to Th. (1,1) they cannot sum to zero. This means that P cannot be an equilibrium point in the spherical field and that consequently Z cannot be an equilibrium point in the corresponding plane field. This contradiction to our assumption concerning Z proves the first part of Laguerre's Theorem.

To prove the second part of the theorem, let us assume first that a circle K through Z and ζ has at least one z_j in its interior, no z_j in its exterior and the remaining z_j on its circumference. The corresponding circle K' through P and Q on the sphere then has at least one P_j in its "interior", no P_j in its "exterior" and the remaining P_j on its circumference. The forces Φ_j are then directed from P along the tangent line to K' at P or to one side of this line and hence cannot sum to zero. This contradicts the hypothesis that Z is a zero of $f_1(z)$ and so at least one z_j must be exterior to K. Since a contradiction would also follow if K were assumed to have at least one z_k in its exterior and no z_j in its interior, we conclude that K must separate the z_j unless it passes through all of them.

While the proof which we have just completed was based upon the properties of equilibrium points, our second proof of Laguerre's Theorem (13,1) will be based upon the properties of the centroid of a system of masses. If Z is any zero of (13,2), it satisfies the equation

$$\frac{n}{Z - \zeta} = \sum_{j=1}^{p} \frac{m_j}{Z - z_j}. \tag{13,3}$$

On substituting into this equation

$$\bar{w} = (Z - \zeta)^{-1}, \qquad \bar{w}_i = (Z - z_i)^{-1}, \tag{13,4}$$

we derive the relation

$$w = \left(\sum_{j=1}^{p} m_j w_j \right) / n, \qquad n = \sum_{j=1}^{p} m_j . \tag{13,5}$$

Consequently, w is the centroid of the system of masses m_j at the points w_j.

As to the location of the centroid w, we have the

LEMMA (13,1). *If each particle w_j in a system of positive masses m_j lies in a circle C', then their centroid w also lies in C' and any line L through w either passes through all the w_j or separates the w_j.*

This lemma is intuitively obvious. In order to prove it analytically, let us write eq. (13,5) as

$$m_1(w_1 - w) + m_2(w_2 - w) + \cdots + m_p(w_p - w) = 0. \tag{13,6}$$

If circle C' did not contain w, it would subtend in w an angle A, $0 < A < \pi$, in which would lie all the vectors $w_j - w$. By Th. (1,1), therefore, the sum (13,6) could not vanish.

Now, to prove the first part of Laguerre's Theorem, let us assume that point Z is exterior to region C and consequently is different from all the z_j. Using Lemma (12,1), we then infer that each w_j defined by (13,4) lies interior to some circle C'; using Lemma (13,1), we infer that the centroid w also lies in C' and, again using Lemma (12,1), we infer that the ζ, defined by eq. (13,4), must also lie in C. That is, not both Z and ζ may lie exterior to C.

In the second part of Laguerre's Theorem we know by hypothesis that Z is different from all the z_j. Any circle K through Z and ζ would transform into a line L through w, the centroid of the w_j. According to Lemma (13,1), either L passes through all the w_j or L separates some w_j from the remaining w_j. Hence, either K passes through all the z_j or it separates some z_j from the remaining z_j. Thus, we have completed the proof of Laguerre's Theorem.

In our discussion of Laguerre's Theorem, we have implied that ζ is a given point and that the zeros Z of $f_1(Z)$ were to be found. Instead, we may consider Z as an arbitrary given point and then define ζ as the solution of the equation

$$n/(Z - \zeta) = f'(Z)/f(Z). \tag{13,7}$$

Thus ζ may be interpreted as the point at which all the mass must be concentrated in order to produce at Z the same resultant force as the system of masses m_j at the points z_j. That is, ζ may be interpreted as the center of force. Based upon this interpretation, a theorem equivalent to Laguerre's Theorem has been given by Walsh [1b, p. 102].

EXERCISES. Prove the following.

1. Laguerre's Theorem may be derived by assuming Z and ζ as both exterior

to region C, by applying the transformation $w = 1/(z - \zeta)$ and finally by using the Lucas Theorem (6,2).

2. If all the points z_j lie on a circle C, the following is true: (a) Z and ζ may not be both interior or both exterior to C; (b) if Z is on C, ζ is located on C at a point separated from Z by at least one z_j ; (c) if ζ is on C, Z is located on C at a point separated from ζ by at least one z_j .

3. Let z_1 be any zero of an $(n + 1)$th degree polynomial $g(z)$ and Z any zero of its derivative. Then any circle through Z and ζ, where

$$\zeta = Z - n(z_1 - Z),$$

must contain at least one zero of $g(z)$ [Féjer 2]. Hint: Writing $g(z) = (z - z_1)f(z)$, compute $g'(Z)/g(Z)$ in terms of $f'(Z)/f(Z)$ and define ζ as in eq. (13,7).

4. The center of gravity of the zeros of the derivative of a polynomial $f(z)$ is the same as the center of gravity of the zeros of $f(z)$.

5. Let $f(z)$ be an nth degree polynomial, t an arbitrary point for which $f(t)f'(t) \neq 0$, and L an arbitrary line through t. Let H be the half-plane bounded by L and containing the point $u = t - [f(t)/f'(t)]$ and let C be the circle which passes through the points t and $v = t - [pf(t)/f'(t)]$ and is tangent at t to L. Then either C contains at least one zero of $f(z)$ or each zero of $f(z)$ lies on C or on L [Nagy 6].

6. In ex. (13,5), let p of the derivatives $f^{(k)}(z)$, $k = 1, 2, \cdots , n$, be different from zero at $z = t$. Then at least one zero of $f(z)$ lies in each circle through the points t and v [Féjer 2].

7. If $z_1 , z_2 , \cdots , z_n$ are the zeros of an nth degree polynomial $f(z)$ and $z_1' , z_2' , \cdots , z_{n-1}'$ are those of its derivative, then

$$(n - 1)^{-1} \sum_1^{n-1} | \mathfrak{J}(z_j') | \leqq n^{-1} \sum_1^n | \mathfrak{J}(z_j) |$$

with the equality holding if and only if $\mathfrak{J}(z_j) > 0$ or < 0 for all j. [De Bruijn 1; De Bruijn-Springer 1; Erdös-Niven 1]. Hint: If $\mathfrak{J}(z_j) > 0$ for all j, use Th. (6,1) and ex. (13,4). If $\mathfrak{J}(z_j) > 0$ for $j \leqq k$ but $\mathfrak{J}(z_j) < 0$ for $j > k$, apply the same to $f_k(z) = \prod_1^k (z - z_j) \prod_{k+1}^n (z - \bar{z}_j)$, noting that $| f'(x) | \leqq | f_k'(x) |$ for all real x.

14. Successive polar derivatives. Corresponding to a given nth degree polynomial $f(z)$, let us construct the sequence of polar derivatives

$$(14,1) \qquad f_k(z) = (n - k + 1)f_{k-1}(z) + (\zeta_k - z)f_{k-1}'(z), \qquad k = 1, 2, \cdots , n,$$

with $f_0(z) = f(z)$. The poles ζ_k may be equal or unequal.

Like the kth ordinary derivative $f^{(k)}(z)$ of $f(z)$, the kth polar derivative $f_k(z)$ is a polynomial of degree $n - k$. Just as the position of the zeros of $f^{(k)}(z)$ may be determined by repeated application of the Lucas Th. (6,2) (see ex. 6,2), the position of the zeros of $f_k(z)$ may be determined by repeated application of Laguerre's Th. (13,1). The result [Laguerre 1b, Takagi 1] so obtained may be stated as

THEOREM (14,1). *If all the zeros of an nth degree polynomial $f(z)$ lie in a circular region C and if none of the points $\zeta_1, \zeta_2, \cdots, \zeta_k$ ($k \leq n - 1$) lies in region C, then each of the polar derivatives $f_1(z), f_2(z), \cdots, f_k(z)$, in the eqs.* (14,1), *has all of its zeros in region C.*

For, by Laguerre's Theorem (13,1), all the zeros of $f_1(z)$ lie in C; hence, all those of $f_2(z)$ lie in C; hence, all those of $f_3(z)$ lie in C; etc.

Let us express the polar derivative $f_k(z)$ directly in terms of $f(z)$ and ζ_1, $\zeta_2, \cdots, \zeta_k$. If from eqs. (14,1) we successively eliminate $f_1(z), f_2(z), \cdots, f_{k-1}(z)$, we find

$$f_k(z) = \sum_{j=0}^{k} (k - j)!C(n - j, k - j)S_j(z)f^{(j)}(z),$$

where $S_j(z)$ is the sum of all the products of the differences $(\zeta_i - z)$ taken j at a time, $i = 1, 2, \cdots, k$.

As is clear from this formula, $f_k(z)$ is a generalization of the kth derivative of a polynomial in the sense that, as $\zeta_j \to \infty$, $j = 1, 2, \cdots, k$,

$$\lim \frac{f_k(z)}{(\zeta_1 - z)(\zeta_2 - z) \cdots (\zeta_k - z)} = f^{(k)}(z).$$

Let us put $f_k(z)$ in still another form which will more clearly show the relation of its coefficients to those of $f(z)$. For this purpose, let us write $f(z)$ and $f_k(z)$ in the form

$$(14,2) \qquad f(z) = \sum_{j=0}^{n} C(n, j)A_j z^j, \qquad f_k(z) = \sum_{j=0}^{n-k} C(n - k, j)A_j^{(k)} z^j,$$

where we define $A_j^{(k)} = 0$ for $j < 0$ and $j > n - k$.

Substituting into eq. (14,1) the expressions for $f_{k-1}(z)$ and $f_k(z)$, equating the combined coefficient of z^j on the right side of eq. (14,1) to that on the left side and simplifying the resulting formulas, we find

$$(14,3) \qquad A_j^{(k)} = (n - k + 1)(A_j^{(k-1)} + A_{j+1}^{(k-1)}\zeta_k).$$

Let us now show that by repeated application of eq. (14,3) we may derive the formula

$$(14,4) \qquad A_j^{(k)} = n(n - 1) \cdots (n - k + 1) \sum_{i=0}^{k} \sigma(k, i)A_{i+j},$$

where $\sigma(k, i)$ is the symmetric function consisting of the sum of all possible products of $\zeta_1, \zeta_2, \cdots, \zeta_k$ taken i at a time. First we note that for $k = 1$ eq. (14,4) is the same as eq. (14,3). We have merely to show then that, if (14,4) is valid, $A_j^{(k+1)}$ will be given by eq. (14,4) with k replaced by $k + 1$. According to (14,3) and (14,4)

$$A_j^{(k+1)} = n(n-1)\cdots(n-k)\sum_{i=0}^{k}[\sigma(k,i)A_{i+j} + \zeta_{k+1}\sigma(k,i)A_{i+j+1}]$$

$$= n(n-1)\cdots(n-k)\sum_{i=0}^{k}[\sigma(k,i) + \zeta_{k+1}\sigma(k,i-1)]A_{i+j}$$

$$= n(n-1)\cdots(n-k)\sum_{i=0}^{k+1}\sigma(k+1,i)A_{i+j} .$$

Thus eq. (14,4) has been established by mathematical induction.

EXERCISES. Prove the following.

1. If $z_1, z_2, \cdots, z_n, \zeta_1, \zeta_2, \cdots, \zeta_k$ all lie on a circle C, then all the zeros of $f_1(z), f_2(z), \cdots, f_k(z)$ also lie on C. Hint: Use ex. (13,2).

2. Let $C_p : |z - z_0| = r_p$ be the circle on which lie the p roots ζ of the kth derivative of eq. (13,7); viz.,

$$(14,5) \qquad n/(z_0-\zeta)^p = \sum_{k=1}^{n} 1/(z_0-z_k)^p = (-1)^{p-1}F^{(p)}(z_0)/(p-1)!,$$

where $F(z) = f'(z)/f(z)$. Then either at least one zero of $f(z)$ lies inside C_p or all the zeros of $f(z)$ lie on C_p. Hint: Label the zeros z_k in the order of increasing distance from z_0 so that

$$|z_0 - z_1| \leqq |z_0 - z_2| \leqq \cdots \leqq |z_0 - z_n|$$

and study the modulus of the left and middle members of eq. (14,5) [Nagy 6 and 12].

3. Let polar co-ordinates (r, ϕ) be introduced with pole at z_0 and with polar axis along a ray from z_0 through a root ζ of eq. (14,5). Then at least one zero of $f(z)$ lies in the curve with the equation $r^p = r_p^p \cos p\phi$ [Nagy 6,12].

CHAPTER IV

COMPOSITE POLYNOMIALS

15. **Apolar polynomials.** So far we have been concerned with the relative position of the zeros of certain pairs of polynomials. In chapters I and II, the pair consisted of a polynomial and its ordinary derivative. In Chapter III, the pair consisted of a polynomial and its polar derivative. We shall now apply the results obtained to the study of the comparative location of the zeros of other pairs or sets of related polynomials.

We begin with a pair of so-called *apolar* polynomials. Two polynomials

$$f(z) = \sum_{k=0}^{n} C(n, k)A_k z^k, \qquad g(z) = \sum_{k=0}^{n} C(n, k)B_k z^k, \qquad A_n B_n \neq 0, \tag{15,1}$$

are said to be apolar if their coefficients satisfy the equation

$$A_0 B_n - C(n, 1)A_1 B_{n-1} + C(n, 2)A_2 B_{n-2} + \cdots + (-1)^n A_n B_0 = 0. \tag{15,2}$$

Clearly, there are an infinite number of polynomials which are apolar to a given polynomial. For example, the polynomial $z^3 + 1$ is apolar to the polynomial $z^3 + 3\alpha z^2 + 3\beta z + 1$ for any choice of the constants α and β.

Let us denote by $z_1, z_2, \cdots, z_n$ the zeros of $f(z)$ and by $\zeta_1, \zeta_2, \cdots, \zeta_n$ the zeros of $g(z)$ so that

$$\begin{aligned} f(z) &= A_n(z - z_1)(z - z_2) \cdots (z - z_n), \\ g(z) &= B_n(z - \zeta_1)(z - \zeta_2) \cdots (z - \zeta_n). \end{aligned} \tag{15,3}$$

In terms of the elementary symmetric functions

$$\begin{aligned} s(n, p) &= \sum z_{j_1} z_{j_2} \cdots z_{j_p}, \\ \sigma(n, p) &= \sum \zeta_{j_1} \zeta_{j_2} \cdots \zeta_{j_p}, \end{aligned} \tag{15,4}$$

the sum of products of these zeros taken p at a time, we may substitute

$$\begin{aligned} C(n, p)A_{n-p} &= (-1)^p s(n, p)A_n, \\ C(n, p)B_{n-p} &= (-1)^p \sigma(n, p)B_n \end{aligned} \tag{15,5}$$

into eq. (15,2) and so obtain the following criterion for apolarity.

THEOREM (15,1). *Two nth degree polynomials $f(z)$ and $g(z)$ are apolar if and only if the elementary symmetric functions $s(n, p)$ of the zeros of $f(z)$ and the elementary symmetric functions $\sigma(n, p)$ of the zeros of $g(z)$ satisfy the relation:*

$$\sum_{k=0}^{n} (-1)^k [C(n, k)]^{-1} s(n, n-k)\sigma(n, k) = 0. \tag{15,6}$$

A simple method for constructing a polynomial $g(z)$ apolar to a given polynomial $f(z)$ is described in Szegö [1], as follows.

THEOREM (15,2). *If the coefficients $a_k = C(n, k)A_k$ of the polynomial $f(z)$ satisfy the linear relation*

$$L[f(t)] = \sum_{k=0}^{n} l_k a_k = 0, \tag{15,7}$$

then the polynomial

$$g(z) = L[(t-z)^n] \tag{15,8}$$

is apolar to $f(z)$.

For, as

$$(t-z)^n = \sum_{k=0}^{n} (-1)^{n-k} C(n, k) z^{n-k} t^k$$

and

$$g(z) = L[(t-z)^n] = \sum_{k=0}^{n} (-1)^{n-k} l_k C(n, k) z^{n-k},$$

eq. (15,7) is seen to be of the form (15,2) with $B_{n-k} = (-1)^{n-k} l_k$.

Now, as to the relative location of the zeros of two apolar polynomials, we have the fundamental result of Grace [1], also proved in Kakeya [3], Szegö [1], Cohn [1], Curtiss [1], Egerváry [1] and Dieudonné [4].

GRACE'S THEOREM (Th. 15,3). *If $f(z)$ and $g(z)$ are apolar polynomials and if one of them has all its zeros in a circular region C, then the other will have at least one zero in C.*

Let us prove this theorem on the assumption that all the zeros $z_1, z_2, \cdots, z_n$ of $f(z)$ lie in a circular region C. If the zeros $\zeta_1, \zeta_2, \cdots, \zeta_{n-1}$ of $g(z)$ were all to lie exterior to C, all the zeros of each polar derivative $f_k(z)$, $k = 1, 2, \cdots, n-1$, given by eqs. (14,1) would according to Th. (14,1) also lie in C.

In particular, let us consider $f_{n-1}(z)$ which according to eqs. (14,2) and (14,4) we may write as

$$f_{n-1}(z) = A_0^{(n-1)} + A_1^{(n-1)} z, \tag{15,9}$$

where

$$A_0^{(n-1)} = n!\{A_0 + \sigma(n-1, 1)A_1 + \sigma(n-1, 2)A_2 + \cdots + \sigma(n-1, n-1)A_{n-1}\},$$

$$A_1^{(n-1)} = n!\{A_1 + \sigma(n-1, 1)A_2 + \sigma(n-1, 2)A_3 + \cdots + \sigma(n-1, n-1)A_n\}.$$

In view of eqs. (15,4) and (15,9) and the relation

$$\sigma(n-1, k) + \sigma(n-1, k-1)\zeta_n = \sigma(n, k),$$

it follows that

$$f_{n-1}(\zeta_n) = n!\{A_0 + \sigma(n, 1)A_1 + \sigma(n, 2)A_2 + \cdots + \sigma(n, n)A_n\}$$

$$= \frac{n!}{B_n}\{A_0B_n - C(n, 1)A_1B_{n-1} + C(n, 2)A_2B_{n-2} - \cdots + (-1)^nC(n, n)A_nB_0\}. \tag{15,10}$$

Since $f(z)$ and $g(z)$ are apolar, eq. (15,10) implies that $f_{n-1}(\zeta_n) = 0$. The point ζ_n is therefore the zero of $f_{n-1}(z)$ and must lie in C.

In other words, at least one of the zeros $\zeta_1, \zeta_2, \cdots, \zeta_n$ of $g(z)$ must lie in any circular region C containing all the zeros of $f(z)$. Similarly, at least one of the zeros $z_1, z_2, \cdots, z_n$ of $f(z)$ lies in any circular region containing all the zeros of $g(z)$.

From Grace's Theorem, we may deduce at once the following result due to Takagi [1].

COROLLARY (15,3). *If $f(z)$ and $g(z)$ are apolar polynomials, any convex region A enclosing all the zeros of $f(z)$ must have at least one point in common with any convex region B enclosing all the zeros of $g(z)$.*

For, if A and B had no point in common, we could separate them by means of a circle C enclosing say A, but not containing any zero of $g(z)$. This would contradict Grace's Theorem.

From Grace's Theorem, we may also infer the following result due to Walsh [6].

THEOREM (15,4). *Let $\Phi(z_1, z_2, \cdots, z_n)$ be a linear symmetric function in the variables $z_1, z_2, \cdots, z_n$ and let C be a circular region containing the n points $z_1^{(0)}, z_2^{(0)}, \cdots, z_n^{(0)}$. Then in C there exists at least one point ζ such that $\Phi(\zeta, \zeta, \cdots, \zeta) = \Phi(z_1^{(0)}, z_2^{(0)}, \cdots, z_n^{(0)})$.*

For, if $\Phi(z_1^{(0)}, z_2^{(0)}, \cdots, z_n^{(0)}) = \Phi_0$, the differences $\Phi(z_1, z_2, \cdots, z_n) - \Phi_0$ is linear and symmetric in the $z_1, z_2, \cdots, z_n$. By the well-known theorem of algebra, any function linear and symmetric in the variables $z_1, z_2, \cdots, z_n$ may be expressed as a linear combination of the elementary symmetric functions $s(n, p)$ of these variables. That is, we may find constants B_k so that

$$\Phi(z_1, z_2, \cdots, z_n) - \Phi_0 = B_0 s(n, 0) + B_1 s(n, 1) + \cdots + B_n s(n, n)$$

$$= A_n^{-1}\{B_0 A_n - C(n, 1)B_1 A_{n-1} + \cdots$$

$$+ (-1)^n C(n, n) B_n A_0\},$$

where $f(z) = \prod_{j=1}^n (z - z_j) = \sum_{j=0}^n C(n, j) A_j z^j$. Consequently,

$$\Phi(z_1^{(0)}, z_2^{(0)}, \cdots, z_n^{(0)}) - \Phi_0 = 0$$

is a relation of type (15,2) and by Th. (15,2), $f(z)$ is apolar to the polynomial

$$g(z) = \sum_{k=0}^n C(n, k) B_k z^k = \Phi(z, z, \cdots, z) - \Phi_0 .$$

By Th. (15,3), $g(z)$ must have at least one zero ζ in C.

Conversely, as may be shown by a reversal of the above steps, Th. (15,4) implies Th. (15,3). In other words, as shown in Curtiss [1], Th. (15,3) and Th. (15,4) are equivalent theorems.

Exercises. Prove the following.

1. If $2n - 1$ of the $2n$ points $z_1, z_2, \cdots, z_n, \zeta_1, \zeta_2, \cdots, \zeta_n$ lie on a circle C, then also the remaining point lies on C.

2. Let $F(z) = \sum_{k=0}^n C(n, k)\alpha_k z^k$ and $G(z) = \sum_{k=0}^n C(n, k)\beta_k z^k$ satisfy the relation $\sum_{k=0}^n (-1)^k C(n, k)\alpha_k \bar{\beta}_k = 0$. If all the zeros of $F(z)$ lie in a circular region K, then at least one zero of $G(z)$ lies in the circular region K' obtained on inverting K in the unit circle. Hint: Apply Th. (15,3) to

$$f(z) = z^n \bar{F}(1/z) \text{ and } g(z) = G(z).$$

3. If $f(z)$ and $g(z)$ are apolar polynomials with only real zeros, any interval A containing the zeros of $f(z)$ must have at least one point in common with any interval B containing the zeros of $g(z)$. Hint: Use Cor. (15,3).

16. **Some applications.** We shall now apply Ths. (15,3) and (15,4) to the study of polynomials $h(z)$ which are derived in various ways by the composition of two given polynomials $f(z)$ and $g(z)$. In this regard we shall first consider a result due to Szegö [1].

Theorem (16,1). *From the given polynomials*

$$f(z) = \sum_{k=0}^n C(n, k) A_k z^k, \qquad g(z) = \sum_{k=0}^n C(n, k) B_k z^k, \tag{16,1}$$

let us form the third polynomial

$$h(z) = \sum^n C(n, k) A_k B_k z^k. \tag{16,2}$$

If all the zeros of $f(z)$ lie in a circular region A, then every zero γ of $h(z)$ has the form $\gamma = -\alpha\beta$ where α is a suitably chosen point in A and β is a zero of $g(z)$.

This follows from Th. (15,3). For, since the equation

$$h(\gamma) = \sum_{k=0}^{n} C(n, k)A_kB_k\gamma^k = 0$$

defines a linear relation $L[f(t)] = 0$ among the coefficients of $f(z)$, the polynomial

$$L[(t - z)^n] = \sum_{k=0}^{n} (-1)^k C(n, k)B_k\gamma^k z^{n-k} = z^n g(-\gamma/z)$$

is apolar to $f(z)$ and thus has at least one zero α in A. If the zeros of $g(z)$ are denoted by $\beta_1, \beta_2, \cdots, \beta_n$, the zeros of $z^n g(-\gamma/z)$ will be $-\gamma/\beta_1, -\gamma/\beta_2, \cdots, -\gamma/\beta_n$. One of these will be α. That is, $\gamma = -\alpha\beta_j$ for some j.

Theorem (16,1) leads at once to the following result of Cohn [1] and Egeváry [1].

COROLLARY (16,1). *If all the zeros of $f(z)$ lie in the circle $|z| < r$ and if all the zeros of $g(z)$ lie in the circle $|z| \leqq s$, then all the zeros of $h(z)$ of eq.* (16,2) *lie in the circle $|z| < rs$.*

For, by hypothesis $|\alpha| < r$ and $|\beta| \leqq s$ and thus $|\gamma| = |\alpha\beta| < rs$.

The next two theorems, which are due to Marden [12], deal with a different variety of composite polynomials than those treated in Th. (16,1). They are generalizations of the results stated in exs. (16,4), (16,5), (16,6) and (16,7).

THEOREM (16,2). *From the given polynomials*

$$f(z) = \sum_{k=0}^{m} a_k z^k, \qquad g(z) = \sum_{k=0}^{n} b_k z^k, \tag{16,3}$$

let us form the polynomial

$$h(z) = \sum_{k=0}^{m} a_k g(k) z^k. \tag{16,4}$$

If all the zeros of $f(z)$ lie in the ring R_0

$$R_0: \quad 0 \leqq r_1 \leqq |z| \leqq r_2 \leqq \infty, \tag{16,5}$$

and if all the zeros of $g(z)$ lie in the annular region A

$$A: \quad 0 \leqq \rho_1 \leqq |z|/|z - m| \leqq \rho_2 \leqq \infty, \tag{16,6}$$

then all the zeros of $h(z)$ lie in the ring R_n

$$R_n: \quad r_1 \min(1, \rho_1^n) \leqq |z| \leqq r_2 \max(1, \rho_2^n). \tag{16,7}$$

It is to be observed that the region A has as boundary curves the circles

$$|z_1| = \rho_1 |z - m| \qquad \text{and} \qquad |z_2| = \rho_2 |z - m|,$$

each of which is the locus of a point which moves so that its distance from the origin is a constant times its distance from the point $z = m$. The region A in

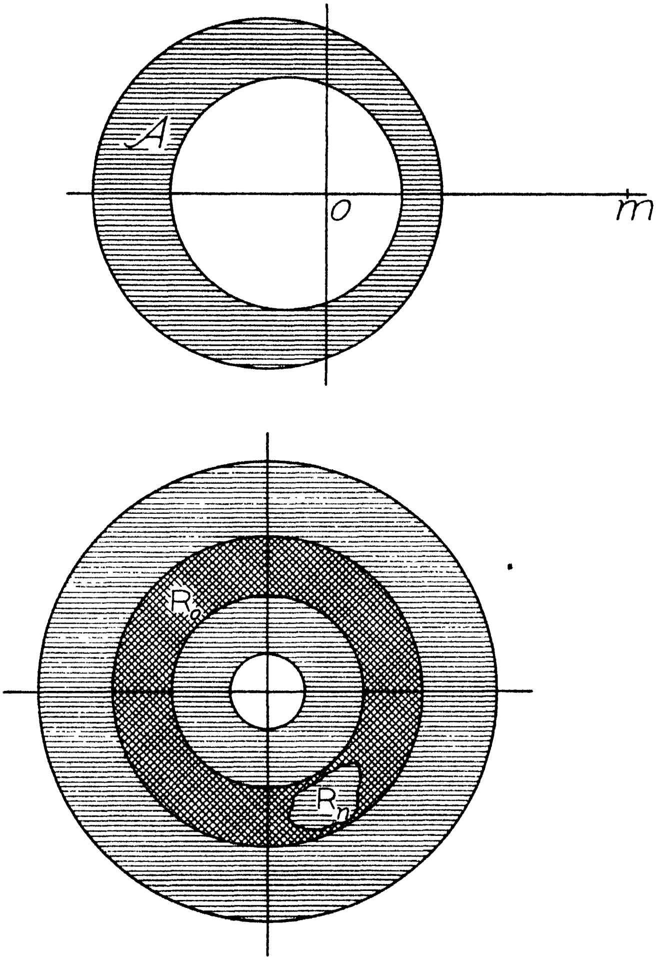

FIG. (16, 1)

Fig. (16,1) typifies the case $0 < \rho_1 < \rho_2 < 1$. We leave to the reader to sketch A in the cases $0 < \rho_1 < 1 < \rho_2$ and $1 < \rho_1 < \rho_2$, as well as in the cases in which either ρ_1 or ρ_2 or both assume the values 0,1 or ∞.

To prove Th. (16,2), we shall need

LEMMA (16,2a). *If $\beta_1 \neq m$ and if all the zeros of $f(z)$ lie in a circular region C, then every zero Z of the polynomial*

$$f_1(z) = -zf'(z) + \beta_1 f(z) \tag{16,8}$$

may be written in the form $Z = \zeta$ *or in the form*

$$Z = [\beta_1/(\beta_1 - m)]\zeta \tag{16,9}$$

where ζ *is a point of* C.

This lemma follows from Th. (15,4). For, since eq. $f_1(Z) = 0$ is linear and symmetric in the zeros of $f(z)$, there exists in C a point ζ such that

$$0 = f_1(Z) = -mZ(Z - \zeta)^{m-1} + \beta_1(Z - \zeta)^m,$$

whence $Z = \zeta$ or Z has the form (16,9).

We shall need also the

LEMMA (16,2b). *If* β_1 *is a zero of* $g(z)$ *and if the hypotheses of Th.* (16,2) *are satisfied, then all the zeros of the* $f_1(z)$ *in* (16,8) *lie in the ring* R_1 ,

$$R_1: \quad r_1 \min(1, \rho_1) \leqq |z| \leqq r_2 \max(1, \rho_2). \tag{16,10}$$

By the hypotheses of this lemma,

$$\rho_1 \leqq |\beta_1|/|\beta_1 - m| \leqq \rho_2 . \tag{16,11}$$

Since all the zeros of $f(z)$ lie in the region $|z| \leqq r_2$, it follows from Lem. (16,2a) that

$$|\zeta| \leqq r_2 . \tag{16,12}$$

Either $Z = \zeta$, whereupon $|Z| \leqq r_2$, or Z has form (16,9) whereupon $|Z| \leqq \rho_2 r_2$. Hence, $|Z| \leqq \max(1, \rho_2)r_2$. Similarly, since all the zeros of $f(z)$ lie in the region $|z| \geqq r_1$, it follows by use of Lem. (16,2a) that $|Z| \geqq \min(1, \rho_1)r_1$. This verifies Lemma (16,2b).

Finally, we shall need the

LEMMA (16,2c). *Let* $\beta_1, \beta_2, \cdots, \beta_n$ *be the zeros of* $g(z)$ *and let* $\{f_k(z)\}$ *be the sequence of polynomials*

$$f_0(z) = f(z), \qquad f_k(z) = \beta_k f_{k-1}(z) - zf'_{k-1}(z), \qquad k = 1, 2, \cdots, n. \tag{16,13}$$

If $f(z)$ *and* $g(z)$ *satisfy the hypothesis of Th.* (16,2), *then all the zeros of* $f_n(z)$ *lie in the ring* R_n .

This lemma is clearly true for $n = 1$, for then it is identical with Lem. (16,2b). Let us assume its validity for $n = k - 1$; i.e., that the zeros of $f_{k-1}(z)$ lie in the ring R_{k-1} :

$$r_1' = r_1 \min(1, \rho_1^{k-1}) \leqq |z| \leqq r_2 \max(1, \rho_2^{k-1}) = r_2' . \tag{16,14}$$

Applying Lemma (16,2b) with r_1 and r_2 replaced by r_1' and r_2', we find that all the zeros of $f_k(z)$ lie in the ring R_k since

$$r_1' \min(1, \rho_1) = r_1 \min(1, \rho_1^k), \qquad r_2' \max(1, \rho_2) = r_2 \max(1, \rho_2^k).$$

That is, Lem. (16,2c) has been established by mathematical induction.

Now, to prove Th. (16,2), we have only to show that $f_n(z)$ is essentially $h(z)$. For this purpose, let us define

$$g_k(z) = b_n(\beta_1 - z)(\beta_2 - z) \cdots (\beta_k - z), \qquad b_n \neq 0,$$

and compute $f_1(z)$ from eqs. (16,13) as

$$f_1(z) = \sum_{j=0}^{m} a_j(\beta_1 - j)z^j = b_n^{-1} \sum_{j=0}^{m} a_j g_1(j) z^j.$$

If we now assume that

$$f_{k-1}(z) = b_n^{-1} \sum_{j=0}^{m} a_j g_{k-1}(j) z^j, \tag{16,15}$$

we may compute $f_k(z)$ from eqs. (16,13) as

$$f_k(z) = b_n^{-1}\left\{ \sum_{j=0}^{m} a_j g_{k-1}(j)(\beta_k - j) z^j \right\} = b_n^{-1} \sum_{j=0}^{m} a_j g_k(j) z^j.$$

In other words, $h(z) = b_n f_n(z)$ and thus by Lemma (16,2c) all the zeros of $h(z)$ lie in the ring R_n, as was to be proved.

Another theorem of Marden [12] involving the same polynomials $f(z)$, $g(z)$ and $h(z)$ as in Th. (16,2) is

THEOREM (16,3). *Let $f(z)$, $g(z)$ and $h(z)$ be the polynomials defined in Th. (16,2). If all the zeros of $f(z)$ lie in the sector S_0 :*

$$\omega_1 \leqq \arg z \leqq \omega_2 , \qquad \omega_2 - \omega_1 = \omega < \pi , \tag{16,16}$$

and if all the zeros of $g(z)$ lie in the lune $\mathfrak{L}$:

$$\theta_1 \leqq \arg [z/(z - m)] \leqq \theta_2 , \qquad | \theta_1 | + | \theta_2 | \leqq (\pi - \omega)/n, \tag{16,17}$$

then all the zeros of $h(z)$ lie in the sector S_n :

$$\omega_1 + \min (0, n\theta_1) \leqq \arg z \leqq \omega_2 + \max (0, n\theta_2). \tag{16,18}$$

Here the boundary curves of $\mathfrak{L}$,

$$\arg z/(z - m) = \theta_1 \qquad \text{and} \qquad \arg z/(z - m) = \theta_2 ,$$

are the arcs of circles, each of which is the locus of a point in which the line-segment $z = 0$ to $z = m$ subtends a constant angle θ. The region $\mathfrak{L}$ in Fig. (16,2) typifies the case that $\theta_1 < \theta_2 < 0$. We leave to the reader to sketch $\mathfrak{L}$ in the cases $\theta_1 < 0 < \theta_2$ and $0 < \theta_1 < \theta_2$ as well as in the special cases when either or both θ_1 and θ_2 are 0 or π.

The proof of this theorem is similar to that of Th. (16,2), except that the argument of the Z in eq. (16,9) instead of its modulus is used. The necessary lemmas paralleling Lem. (16,2b) and (16,2c) are given in ex. (16,7).

EXERCISES. Prove the following.

1. Th. (16,1) is valid if A is assumed to be an arbitrary convex region. Hint: Use Cor. (15,3).

2. If $f(z)$ has only real zeros and $g(z)$ has only real zeros of like sign, then the $h(z)$ of eq. (16,2) has only real zeros. Hint: Use ex. (16,1).

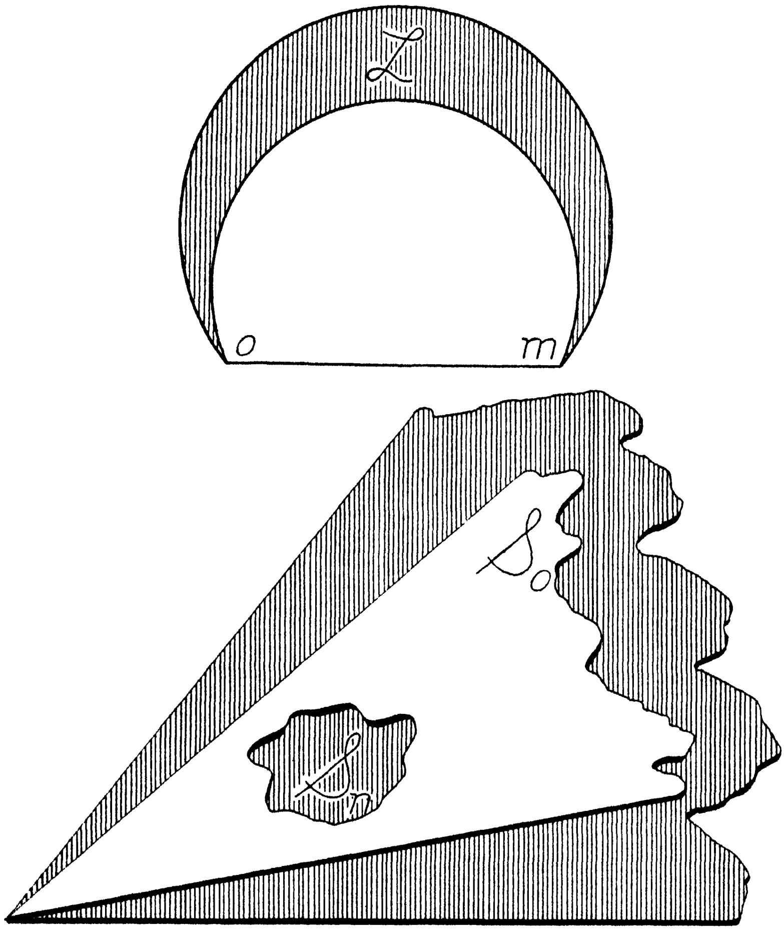

Fig. (16, 2)

3. If $f(z)$ has only real zeros with a sign of ϵ and $g(z)$ have only real zeros with a sign ϵ', then the $h(z)$ of eq. (16,2) has only real zeros of sign $(-\epsilon\epsilon')$ [Takagi 1].

4. If $f(z)$ has zeros only in the sector $\theta \leqq \arg z \leqq \theta'$, where $0 \leqq \theta' - \theta < \pi$ and $g(z)$ has zeros only in the sector $\phi \leqq \arg z \leqq \phi'$ where $0 \leqq \phi' - \phi < \pi$, then the $h(z)$ of eq. (16,2) has zeros only in the sector $\theta + \phi - \pi \leqq \arg z \leqq \theta' + \phi' - \pi$ [Takagi 1].

5. If $f(z)$ has zeros only in the above sector $\theta \leqq \arg z \leqq \theta'$ and $g(z)$ has only real zeros, then the $h(z)$ of (16,2) has all its zeros in the double sector $\theta \leqq \arg (\pm z) \leqq \theta'$.

6. The theorems in the above exs. 2 to 5 remain valid when $h(z)$ is replaced by either

$$h_1(z) = \sum_{k=0}^{n} k![C(n, k)A_k][C(n, k)B_k]z^k$$

or

$$h_2(z) = \sum_{k=0}^{n} [C(n, k)A_k][C(n, k)B_k]z^k.$$

Hint: Use Cor. (18,2c). The results thereby obtained are due to the following: $h_1(z)$: ex. 2, Schur [2]; exs. 3 and 4, Takagi [1]; ex. 5, Takagi [1] and Weisner [3]; $h_2(z)$: ex. 2, Malo [1]; ex. 4 [DeBruijn 3]; ex. 5, Weisner [3].

7. If the hypotheses of Th. (16,3) are satisfied, all the zeros of the $f_1(z)$ of eq. (16,8) lie in the sector $\mathcal{S}_1$. By induction, all the zeros of the $f_n(z)$ of eq. (16,13) lies in $\mathcal{S}_n$ [Marden 12].

8. If all the zeros of $f(z)$ lie in the circle $|z| \leqq r_2$ and if all the zeros of $g(z)$ lie in the half-plane bounded by the perpendicular bisector of segment $z = 0$ to $z = m$ and containing the origin, then all the zeros of the $h(z)$ of eq. (16,4) also lie in the circle $|z| \leqq r_2$ [Obrechkoff 7, Weisner 4]. Hint: Set $r_1 = \rho_1 = 0$ and $\rho_2 = 1$ in Th. (16,2).

9. If all the zeros of $f(z)$ lie exterior to the circle $|z| = r_1$ and all the zeros of $g(z)$ lie in the half-plane $\Re(z) \geqq m/2$, then all the zeros of the $h(z)$ of eq. (16,4) lie exterior to the circle $|z| = r_1$. If all the zeros of $f(z)$ lie on the circle $|z| = r_1$ and those of $g(z)$ lie on the line $\Re(z) = m/2$, then all the zeros of $h(z)$ lie on the circle $|z| = r_1$ [Obrechkoff 7, Weisner 4].

10. If all the zeros of $f(z)$ are real and positive and if all the zeros of $g(z)$ are real and exterior to the interval $0 \leqq z \leqq m$, then all the zeros of $h(z)$ of eq. (16,4) are real and positive [Laguerre 1, pp. 200-202; Pólya 6].

11. If in Th. (16,2) $g(z) = (\beta_1 - z) \cdots (\beta_n - z)$ and if all the zeros of $f(z)$ lie in the ring $r_1 \leqq |z| \leqq r_2$, then all the zeros of $h(z)$ lie in the ring

$$(16,19) \qquad r_1 \min [1, |g(0)/g(m)|] \leqq |z| \leqq r_2 \max [1, |g(0)/g(m)|]$$

[Marden 14].

12. Let $f(z)$, $g(z)$ and $h(z)$ be defined by eqs. (16,1) and (16,2). Let K denote a circle or straight line and K_I and K_E the two closed regions bounded by K. Let $\alpha_1, \alpha_2, \cdots, \alpha_p$ $(p \leqq n)$ denote the zeros of $f(z)$ in K_I and $\alpha_{p+1}, \alpha_{p+2}, \cdots, \alpha_n$ those not in K_I. Let further

$$f_I(z) = A_n \prod_{j=1}^{p} (z - \alpha_j) \prod_{j=p+1}^{n} (z - \alpha_j^*)[(\kappa - \alpha_j)/(\kappa - \alpha_j^*)],$$

$$f_E(z) = A_n \prod_{j=1}^{p} (z - \alpha_j^*)[(\kappa - \alpha_j)/(\kappa - \alpha_j^*)] \prod_{j=p+1}^{n} (z - \alpha_j)$$

where α_j^* denotes the image of point α_j in K and κ denotes an arbitrary but fixed

point on K. Then

(a) all the zeros of $f_I(z)$ lie in K_I and all those of $f_E(z)$ in K_E;

(b) $|f_I(z)| = |f_E(z)| = |f(z)|$ on K;

(c) $|f_I(z)| \geqq |f(z)|$ in K_E and $|f_E(z)| \geqq |f(z)|$ in K_I

[De Bruijn-Springer 2].

13. If $\{f, g\}$ denotes the left-side of eq. (15,2), then, in the notation of ex. (16,12), $|\{f, g\}| \leqq |\{f_E, g_I\}|$. Hint: Using Grace's Theorem, show that $\{f_E - \lambda f, g_I - \lambda g\} \neq 0$ for all $|\lambda| < 1$ [De Bruijn-Springer 2].

1,4. Let $D(z,L)$ denote the distance of point z to line L and let $\alpha_1, \alpha_2, \cdots, \alpha_n$; $\beta_1, \beta_2, \cdots, \beta_n$; $\gamma_1, \gamma_2, \cdots, \gamma_n$ denote respectively the zeros of the $f(z)$, $g(z)$ and $h(z)$ defined by eqs. (16,1) and (16,2). Then

$$\sum_{j=1}^{n} [D(\gamma_j, L) - D(0, L)] \leqq (B_{n-1}/B_n) \sum_{j=1}^{n} [D(\alpha_j, L) - D(0, L)]$$

[De Bruijn-Springer 2].

15. Let $\phi(x) = \max(1, |x|)$. Then, for the α_j, β_j and γ_j of ex. (16,14) and for $r_1 > 0$ and $r_2 > 0$,

$$\prod_{j=1}^{n} \phi(r_1, r_2/\gamma_j) \leqq \prod_{j=1}^{n} \phi(r_1/\alpha_j)\phi(r_2/\beta_j).$$

Hint: Take the K of ex. (16,12) as the circle $|z| = r$; apply to $f_E(z)$, $g_E(z)$ and $h_E(z)$ the *Jensen Formula*

$$\int_0^{2\pi} \log |f(re^{i\theta})/f(0)| \, d\theta = 2\pi \sum_{j=1}^{n} \log \phi(r/\alpha_j),$$

and use exs. (16,12) and (16,13), noting that $h(u) = \{f(uz), z^n g(-1/z)\}$ [DeBruijn-Springer 2].

17. **Linear combinations of polynomials.** Our next application of the theorems of section 15 will be to linear combinations of the polynomials

$$f_k(z) = z^{n_k} + a_{k1}z^{n_k-1} + \cdots + a_{kn_k}, \qquad k = 1, 2, \cdots, p. \tag{17,1}$$

We shall assume that the zeros of $f_k(z)$ lie in a circular region C_k. Unless otherwise specified, the region C_k will be bounded by a circle C_k with center c_k and radius r_k. Our general result is embodied in the

THEOREM (17,1). *The zeros of the linear combination*

$$F(z) = \lambda_1 f_1(z) + \lambda_2 f_2(z) + \cdots + \lambda_p f_p(z), \tag{17,2}$$

where $\lambda_j \neq 0$, $j = 1, 2, \cdots, p$, *lie in the locus* Γ *of the roots of the equation*

$$\lambda_1(z - \alpha_1)^{n_1} + \lambda_2(z - \alpha_2)^{n_2} + \cdots + \lambda_p(z - \alpha_p)^{n_p} = 0 \tag{17,3}$$

when the $\alpha_1, \alpha_2, \cdots, \alpha_p$ *vary independently over the regions* $C_1, C_2, \cdots, C_p$ *respectively.*

This result follows almost at once from Th. (15,4). For, if ζ is any zero of $F(z)$, the corresponding equation $F(\zeta) = 0$ is linear and symmetric in the zeros of each $f_j(z)$. On the strength of Th. (15,4), eq. $F(\zeta) = 0$ may be replaced by an equation for ζ in which all the zeros of each $f_j(z)$ are made to coincide at a suitably chosen point α_j in the region C_j. This leads to eq. (17,3) for ζ. To find all possible positions of ζ, we must allow each α_j to occupy all possible positions in its circular region C_j. In other words, all the zeros of $F(z)$ lie in the locus Γ as defined in Th. (17,1).

It is to be noted that in Th. (17,1) the regions C_j may be half-planes as well as the interior or exterior of circles.

The particular case $p = 2$ and $n_1 = n_2 = n$ is one in which we can readily determine Γ. For that case we write $\lambda_2/\lambda_1 = -\lambda$ and denote by $\omega_1, \omega_2, \cdots, \omega_n$, the nth roots of λ with $\omega_1 = 1$ when $\lambda = 1$. Eq. (17,3) is the same as the equations

$$(17,4) \qquad (z - \alpha_1) - \omega_k(z - \alpha_2) = 0, \qquad k = 1, 2, \cdots, n,$$

whose roots are

$$(17,5) \qquad z_k = \frac{\alpha_1 - \omega_k\alpha_2}{1 - \omega_k},$$

where $k = 1, 2, \cdots, n$ when $\lambda \neq 1$ and $k = 2, 3, \cdots, n$ when $\lambda = 1$. The locus Γ will then consist of the ensemble of loci Γ_k of the z_k when α_1 and α_2 vary over their circular regions C_1 and C_2 respectively.

In order to find Γ_k, we shall need three lemmas which essentially concern the location of the centroid of a system of particles possessing real or complex masses. The first lemma is due to Walsh [1c, pp. 60-61] and [6, p. 169]. All three lemmas are proved in Marden [10].

LEMMA (17,2a). *If the points $\alpha_1, \alpha_2, \cdots, \alpha_p$ vary independently over the closed interiors of the circles $C_1, C_2, \cdots, C_p$ respectively, then the locus of the point α,*

$$(17,6) \qquad \alpha = \sum_{j=1}^{p} m_j\alpha_j,$$

where the m_j are arbitrary complex constants, will be the closed interior of a circle C of center c and radius r, where

$$(17,7) \qquad c = \sum_{j=1}^{p} m_j c_j, \qquad r = \sum_{j=1}^{p} |m_j| r_j$$

and c_j and r_j denote respectively the center and radius of the circle C_j.

In the case of exclusively positive real m_j, we may deduce Lemma (17,2a) from the theorem of Minkowski [1] which states that *the convex point-set K whose envelope-function* (Stützfunktion) *is $H = \sum_{j=1}^{p} m_j H_j$ is the locus of the point $\alpha = \sum_{j=1}^{p} m_j\alpha_j$ when for each $j = 1, 2, \cdots, p$ the point α_j has as locus the convex point-set K_j whose envelope-function is H_j.* For, on taking $K_j = C_j$

and setting $c_j' = \Re(c_j)$, $c_j'' = \Im(c_j)$, $c' = \Re(c)$ and $c'' = \Im(c)$, we find that, since by definition of H_j the equation $ux + vy = H_j$ must represent the family of lines tangent to C_j ,

$$H_j = c_j'u + c_j''v + r_j(u^2 + v^2)^{1/2}$$

and thus that

$$H = c'u + c''v + r(u^2 + v^2)^{1/2}.$$

To prove Lemma (17,2a) as stated, let us note that

$$| \alpha - c | = \left| \sum_{j=1}^{p} m_j(\alpha_j - c_j) \right| \leqq \sum_{j=1}^{p} | m_j | \, | \alpha_j - c_j | \leqq \sum_{j=1}^{p} | m_j | r_j = r,$$

which means that every point α defined by (17,6) lies in C. Conversely, if α is any point in or on C, we may write

$$\alpha = c + \mu r e^{i\theta}, \qquad 0 \leqq \mu \leqq 1,$$

and associate with this α the points α_j

$$\alpha_j = c_j + \mu(| m_j |/m_j)r_j e^{i\theta}.$$

Each point α_j lies in or on C_j and together they satisfy eq. (17,6). In other words, the locus of the point α of eq. (17,6) is the closed interior of circle C.

We turn next to

LEMMA (17,2b). *If the point α_1 describes the closed exterior of the circle C_1 but the remaining α_j describe the closed interiors of the circles C_j , then the locus of the point α of eq.* (17,6) *is the closed exterior of the circle C of center c and radius r, where*

$$(17,8) \qquad c = \sum_{j=1}^{p} m_j c_j , \qquad r = | m_1 | r_1 - \sum_{j=2}^{p} | m_j | r_j$$

provided in (17,8) $r > 0$, *and is the entire plane if* $r \leqq 0$.

To prove this lemma when $r > 0$, let us note that now

$$| \alpha - c | = \left| \sum_{j=1}^{p} m_j(\alpha_j - c_j) \right| \geqq | m_1 | \, | \alpha_1 - c_1 | - \sum_{j=2}^{p} | m_j | \, | \alpha_j - c_j |$$

$$\geqq | m_1 | r_1 - \sum_{j=2}^{p} | m_j | r_j = r.$$

Conversely, with every point α

$$\alpha = c + \mu r e^{i\theta}, \qquad \mu \geqq 1,$$

which lies on or outside C, we may associate the points α_j defined by the equations

$$m_1(\alpha_1 - c_1) = [| m_1 | r_1 + (\mu - 1)r]e^{i\theta},$$

$$m_j(\alpha_j - c_j) = - | m_j | r_j e^{i\theta}, \qquad j = 2, 3, \cdots, p.$$

These α_j satisfy eq. (17,6). Furthermore, this point α_1 lies on or outside circle C_1, whereas the remaining points α_j lie on their respective circles C_j. That is, every point α of the locus lies in C and every point of C is a point of the locus.

If $r = 0$, the locus C is obviously the entire plane.

If $r < 0$, let us choose a circle C_1' concentric with C_1 of radius r_1' such that

$$r' = | m_1 | r_1' - \sum_{j=2}^{p} | m_j | r_j = 0.$$

This $r_1' > r_1$ and hence the exterior of C_1' is contained in the exterior of C_1 and the locus C' of α corresponding to the circles C_1', C_2, C_3, $\cdots$, C_p is contained in the locus C of α corresponding to the circles C_1, C_2, $\cdots$, C_p. Since C' is the entire plane, so is C.

To complete the discussion of the locus of α, we add

LEMMA (17,2c). *If two or more of the α_j vary over the closed exteriors of their circles C_j, then the locus of point α is the entire plane.*

For example, let us suppose that α_1 varies over the closed exterior of the circle C_1 with center c_1 and radius r_1 and α_2 varies over the closed exterior of C_2 while the remaining α_j vary over the closed interiors of the circles C_j. We may then choose a circle C_2' whose interior lies exterior to C_2 and whose radius r_2' satisfies the inequality

$$| m_1 | r_1 - | m_2 | r_2' - \sum_{j=3}^{p} | m_j | r_j \leqq 0.$$

The locus C' of α corresponding to the exterior of C_1 and the interiors of C_2', C_3, $\cdots$, C_p is by Lemma (17,2b) the entire plane. As C_2' lies exterior to C_2, the locus C contains C' and hence is also the entire plane.

Returning now to discussion of the locus of the points z_k of eq. (17,5), we may on the basis of Lemmas (17,2a) and (17,2b) deduce from Th. (17,1) two theorems, of which the first is due to Walsh [6].

THEOREM (17,2a). *If all the zeros of $f_1(z) = z^n + a_1z^{n-1} + \cdots + a_n$ lie in or on the circle C_1 with center c_1 and radius r_1 and if all the zeros of $f_2(z) = z^n + b_1z^{n-1} + \cdots + b_n$ lie in or on the circle C_2 with center c_2 and radius r_2, then each zero of the polynomial*

$$(17,9) \qquad h(z) = f_1(z) - \lambda f_2(z), \qquad \lambda \neq 1,$$

lies in at least one of the circles Γ_k with center at γ_k and radius ρ_k, where

$$(17,10) \qquad \gamma_k = \frac{c_1 - \omega_k c_2}{1 - \omega_k}, \qquad \rho_k = \frac{r_1 + | \omega_k | r_2}{| 1 - \omega_k |}$$

and where the ω_k $(k = 1, 2, \cdots, n)$ *are the nth roots of* λ. *If* $\lambda = 1$, *the same result holds provided the root* $\omega_k = 1$ *is omitted and provided the closed interiors of* C_1 *and* C_2 *have no point in common.*

THEOREM (17,2b). *If in the notation of Th.* (17,2a) *each zero of* $f_1(z)$ *lies on or outside circle* C_1, *if each zero of* $f_2(z)$ *lies in or on circle* C_2 *and if* $r_1 > r_2 \mid \lambda \mid^{1/n}$, *then each zero of* $h(z)$ *in* eq. (17,9) *lies on or outside at least one of the circles* Γ_k *with center* γ_k *and radius* ρ_k, *where*

$$(17,11) \qquad \gamma_k = \frac{c_1 - \omega_k c_2}{1 - \omega_k}, \qquad \rho_k = \frac{r_1 - \mid \omega_k \mid r_2}{\mid 1 - \omega_k \mid}.$$

So far we have obtained, by use of Lemmas (17,2a) and (17,2b), some results concerning the location of the zeros of the linear combination $h(z)$ given in eq. (17,9). For this same function $h(z)$, we may obtain an altogether different set of results if we write the roots z_k of eqs. (17,4) in the form

$$(17,12) \qquad \frac{(z_k - c_1) - (\alpha_1 - c_1)}{(z_k - c_2) - (\alpha_2 - c_2)} = \omega_k .$$

The new results will be in terms of the ellipse E with points c_1 and c_2 as foci and $\mid r_1 - r_2 \mid$ as major axis, the hyperbola H with c_1 and c_2 as foci and $(r_1 + r_2)$ as transverse axis, and the conic K having $\mid \lambda \mid^{1/n}$ as eccentricity, c_1 as focus and the line $\Re(z) = \kappa$ as directrix, with

$$(17,13) \qquad \kappa = \sigma + r_1 \mid \lambda \mid^{-1/n}.$$

These new results will be embodied in the following three theorems, which are due to Walsh [3c] in the case the parameter $\lambda = 1$ in eq. (17,9) and to Nagy [10] for other values of λ.

First we shall prove

THEOREM (17,3a). *In Th.* (17,2a), *if each zero of* $f_1(z)$ *lies on or outside circle* C_1, *if each zero of* $f_2(z)$ *lies in or on circle* C_2 *and if circle* C_2 *is contained in circle* C_1, *then no zero of the polynomial* $h(z) = f_1(z) - \lambda f_2(z)$, $\mid \lambda \mid \leqq 1$, *lies interior to the ellipse E.*

By the hypothesis of Th. (17,3a), α_1 lies on or outside C_1, α_2 lies in or on C_2 and

$$(17,14) \qquad \mid c_2 - c_1 \mid < r_1 - r_2 .$$

Furthermore, since $\mid \lambda \mid \leqq 1$, also $\mid \omega_k \mid \leqq 1$ for all k. From (17,12) it then follows that

$$(17,15) \qquad 1 \geqq \mid \omega_k \mid \geqq \frac{\mid \alpha_1 - c_1 \mid - \mid z_k - c_1 \mid}{\mid \alpha_2 - c_2 \mid + \mid z_k - c_2 \mid} \geqq \frac{r_1 - \mid z_k - c_1 \mid}{r_2 + \mid z_k - c_2 \mid}$$

and, consequently,

$$\mid z_k - c_1 \mid + \mid z_k - c_2 \mid \geqq r_1 - r_2 > 0.$$

In short, z_k must lie on or outside E.

Next, we shall prove

THEOREM (17,3b). *If each zero of $f_1(z)$ lies in or on the circle C_1, if each zero of $f_2(z)$ lies in or on the circle C_2 and if C_1 and C_2 have no common points, then no zero of the polynomial $h(z) = f_1(z) - \lambda f_2(z)$ may lie interior to H_1 if $|\lambda| \geqq 1$, and none interior to H_2 if $|\lambda| \leqq 1$, H_1 and H_2 being the branches of hyperbola H containing respectively c_1 and c_2.*

In this theorem a point "interior to a branch H_j" of a hyperbola means one from which no real tangents to H_j can be drawn; in contrast, a point outside H_j means one from which two real tangents to H_j can be drawn.

By hypothesis, α_1 lies in or on C_1, α_2 lies in or on C_2 and

$$|c_1 - c_2| > r_1 + r_2. \tag{17,16}$$

If $|\lambda| \geqq 1$ and thus $|\omega_k| \geqq 1$, we find from (17,12)

$$1 \leqq |\omega_k| \leqq \frac{|z_k - c_1| + |\alpha_1 - c_1|}{|z_k - c_2| - |\alpha_2 - c_1|} \leqq \frac{|z_k - c_1| + r_1}{|z_k - c_2| - r_2} \tag{17,17}$$

provided $|z_k - c_2| - r_2 > 0$; that is, provided z_k lies outside C_2. From (17,17), we deduce that

$$|z_k - c_2| - |z_k - c_1| \leqq r_1 + r_2,$$

which means that z_k is on or outside of H_1. Similarly, if $|\lambda| \leqq 1$ and thus $|\omega_k| \leqq 1$, we find from (17,12) that

$$1 \geqq |\omega_k| \geqq \frac{|z_k - c_1| - |\alpha_1 - c_1|}{|z_k - c_2| + |\alpha_2 - c_2|} \geqq \frac{|z_k - c_1| - r_1}{|z_k - c_2| + r_2}. \tag{17,18}$$

This implies that

$$|z_k - c_1| - |z_k - c_2| \leqq r_1 + r_2 \tag{17,19}$$

and therefore that z_k lies on or outside of H_2.

Finally, we shall establish

THEOREM (17,3c). *If each zero of $f_1(z)$ lies in or on the circle C_1, if each zero of $f_2(z)$ lies in a closed half-plane S satisfying the relation $\Re(z) \geqq \sigma > 0$ and having no points in common with C_1, then each zero of $h(z) = f_1(z) - \lambda f_2(z)$, exterior to S, lies on the same side of conic K as its focus c_1.*

By hypothesis, α_1 lies in C_1, α_2 lies in S and $\sigma - \Re(c_1) > r_1$. From eq. (17,12) we have for any point z_k exterior to S and C_1

$$\begin{aligned} |\lambda|^{1/n} = |\omega_k| &\geqq \frac{|z_k - c_1| - |\alpha_1 - c_1|}{|z_k - \alpha_2|} \geqq \frac{|z_k - c_1| - r_1}{|\Re(\alpha_2 - z_k)|} \\ &\geqq \frac{|z_k - c_1| - r_1}{\sigma - \Re(z_k)} \end{aligned} \tag{17,20}$$

and hence,

$$(17,21) \qquad |\lambda|^{1/n}\left\{\frac{r_1}{|\lambda|^{1/n}} + \sigma - \Re(z_k)\right\} \geqq |z_k - c_1|.$$

Since the parenthesis on the left side of (17,21) is the distance of z_k to line $\Re(z) = \kappa$ (see (17,13)), each point z_k lies on the side of conic K that contains the focus c_1.

EXERCISES. Prove the following.

1. If all the zeros of a polynomial $f(z) = a_0 + a_1 z + \cdots + a_n z^n$ lie in a circle $|z| \leqq r$, then all the zeros of $F(z) = f(z) - c$ lie in the circle $|z| \leqq r + |c/a_n|^{1/n}$. Hint: Use Th. (15,4) and in eq. (17,1) take $f_1(z) = f(z)$, $n_2 = 0$ and $p = 2$ [Walsh 6].

2. If all the points α_j are exterior to circle C_1 with center at c_1 and radius r_1 and if all the points β_j are interior to a concentric circle C_2 of radius r_2, $|\lambda|^{1/n} r_2 < r_1$, then no zero of $h(z)$ in (17,9) lies inside the concentric circle Γ of radius $\rho = (r_1 - r_2|\lambda|^{1/n})/(1 + |\lambda|^{1/n})$ [Nagy 10]. Hint: Use Th. (17,2b).

3. If ξ is any zero of $h(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1}$, then at least one zero of $f_1(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + z^n$ lies in any circular region containing all the points $z_k = \xi(1 - e^{i2\pi k/n})$, $k = 0, 1, 2, \cdots, n - 1$. Thus at least one zero of $f_1(z)$ lies in the circle $|z| \leqq 2|\xi|$ and, if n is odd, at least one in the circle $|z| \leqq 2|\xi| \cos(\pi/2n)$. Note: limit is attained by $f_1(z) = (1 + z)^n$ [Szegö 1]. Hint: Apply Th. (17,2b).

4. The trinomial eq. $1 - z - cz^n = 0$ has at least one root in every circular region containing all the points $z_k = 1 - e^{2\pi k i/n}$, $k = 0, 1, \cdots, n - 1$. Thus, it has at least one root in the circle $|z| \leqq 2$; if n is odd, at least one root in the circle $|z| \leqq 2\cos(\pi/2n)$. Hint: Apply ex. (17,3) [Szegö 1].

5. An equivalent statement of the result in ex. 3 is that, if $f_1(z) \neq 0$ in $|z| \leqq R$, then $h(z) \neq 0$ in $|z| \leqq R/2$ if n is even and in $|z| \leqq (R/2)\sec(\pi/2n)$ if n is odd. The example $f_1(z) = (z - R)^n$ shows that these are the best possible limits.

6. Let $f_1(z) = z^n + A_k z^{n-k} + A_{k+1} z^{n-k-1} + \cdots + A_n$ and $h(z) = A_{k+1} z^{n-k-1} + \cdots + A_n$. If $h(z)$ has at least one zero in $|z| \leqq r$, then $f_1(z)$ has at least one zero in $|z| \leqq 2r + (A_k)^{1/k}$. Hint: In ex. (17,2) take $f_2(z) = z^n + A_k z^{n-k}$, and $h(z) = f_1(z) - f_2(z)$ [Nagy 10].

7. Let $r_1, r_2, \cdots r_n$ be any positive numbers and A any complex number such that $r_1 r_2 \cdots r_n = |A|$. Let $C_j : |z - z_j| = r_j$, $j = 1, 2, \cdots, n$, and let $f(z) = (z - z_1)(z - z_2) \cdots (z - z_n)$. Then among the A-points of $f(z)$ (that is, the points where $f(z) = A$) none lies inside or outside all the circles C_j. If $|B| \leqq |A|$, each B-point lies in at least one circle C_j' concentric with C_j and of radius $r_j' = |B/A|^{1/n} r_j$; p B-points lie in any point-set K comprised of the closed interiors of p circles C_j, provided K has no point in common with the closed interiors of the other $n - p$ circles C_j. Hint for last result: Study the variation of the Z-points as Z decreases continuously from A to 0 [Nagy 11].

8. Let $u_1, u_2, \cdots, u_n$ be n distinct points inside a circular region C and let

$v_1, v_2, \cdots, v_n$ be n distinct points outside C. Then the determinant $|(u_j - v_k)^n|$, $j, k = 1, 2, \cdots, n$, cannot vanish. Hint: $f(z) = (z - u_1)(z - u_2) \cdots (z - u_n)$ is, due to the linearity of eq. (15,2) in the A_j, not only apolar to $(z - u_j)^n$ for each j, but also apolar to the polynomial $g(z) = \sum_{j=1}^{n} c_j(z - u_j)^n$ for arbitrary constants c_j. Choose c_j so that $g(v_k) = 0$ for $k = 1, 2, \cdots, n$ [Szegö 1].

18. **Combinations of a polynomial and its derivatives.** We conclude the present chapter with the application of the theorems of sec. 15 to linear and other combinations of a polynomial and its derivatives.

We begin with a theorem due to Walsh [6].

THEOREM (18,1). *Let*

$$f(z) = \sum_{j=0}^{n} a_j z^j = a_n \prod_{j=1}^{n} (z - \alpha_j), \tag{18,1}$$

$$g(z) = \sum_{j=0}^{n} b_j z^j = b_n \prod_{j=1}^{n} (z - \beta_j), \tag{18,2}$$

$$h(z) = \sum_{j=0}^{n} (n - j)!b_{n-j}f^{(j)}(z) = \sum_{j=0}^{n} (n - j)!a_{n-j}g^{(j)}(z). \tag{18,3}$$

If all the zeros of $f(z)$ lie in a circular region A, then all the zeros of $h(z)$ lie in the point set C consisting of the n circular regions obtained by translating A in the amount and direction of the vectors β_j.

To prove this theorem, we shall assume Z to be any zero of $h(z)$; i.e.,

$$h(Z) = \sum_{j=0}^{n} (n - j)!b_{n-j}f^{(j)}(Z) = 0. \tag{18,4}$$

Since eq. (18,4) is a linear expression in the coefficients of $f(z)$, we infer from Th. (15,2) that $f(z)$ is apolar to the polynomial obtained on replacing $f(z)$ in eq. (18,4) by $(Z - z)^n$; that is,

$$\sum_{j=0}^{n} (n - j)!b_{n-j}d^{(j)}(Z - z)^n/(dZ)^j = n!g(Z - z).$$

According to Th. (15,3), at least one of the n zeros $Z - \beta_j$ of $g(Z - z)$ must lie in the circular region A containing the zeros of $f(z)$. That is, $Z = \alpha + \beta_j$, where α is a point of A.

An interesting special case under Th. (18,1) is the one in which

$$g(z) = z^{n-1}(z - n\lambda_1) \tag{18,5}$$

and thus

$$h(z) = n!f(z) - (n\lambda_1)(n - 1)!f'(z).$$

Since in this case $\beta_1 = n\lambda_1$, and $\beta_2 = \beta_3 = \cdots = \beta_n = 0$, we obtain the following result found in Fujiwara [2], and Marden [3] and [10].

COROLLARY (18,1). *If all the zeros of an nth degree polynomial $f(z)$ lie in a circular region A, all the zeros of the linear combination*

$$f_1(z) = f(z) - \lambda_1 f'(z) \tag{18,6}$$

lie in the point-set comprised of both A and $A' = T(A, n\lambda_1)$, A' being the region obtained on translating A in the magnitude and direction of the vector $(n\lambda_1)$.

When used in conjunction with Cor. (15,3), the apolarity of $f(z)$ and $g(Z - z)$, which led to Th. (18,1), permits us to infer that any convex region A containing all the zeros of $f(z)$ must overlap every convex region B' containing the zeros of $g(Z - z)$. Since B' may be considered as the locus of the point $z = Z - \beta$ when β varies over a convex region B containing the zeros of $f(z)$, each zero Z of $h(z)$ is expressible in the form $Z = \alpha + \beta$ where α and β are points of A and B respectively. In other words, the following result of Takagi [1] has been proved.

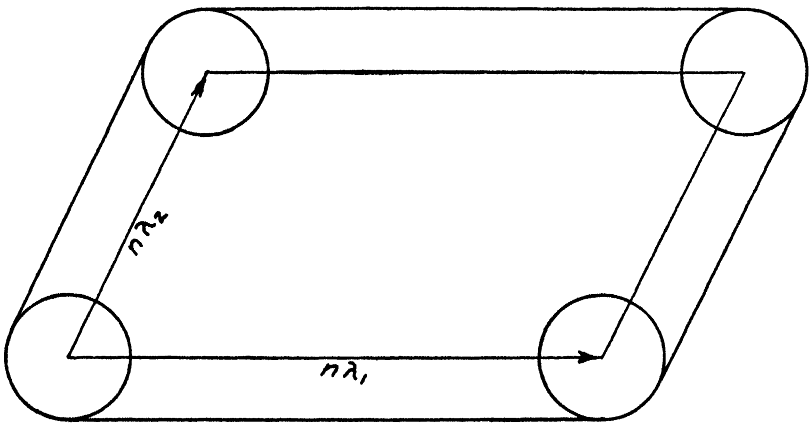

FIG. (18,1)

THEOREM (18,2). *Let the polynomials $f(z)$, $g(z)$ and $h(z)$ be defined by the eqs.* (18,1), (18,2) *and* (18,3). *Then if all the zeros of $f(z)$ lie in a convex region A and all the zeros of $g(z)$ lie in a convex region B, then all the zeros of $h(z)$ lie in the convex region C which is the locus of the points $\gamma = \alpha + \beta$ when the points α and β vary independently over the regions A and B respectively.*

If $g(z)$ is taken as the polynomial (18,5), the convex region B may be taken as the line segment joining the points $z = 0$ and $z = n\lambda_1$. We thereby derive a result due to Takagi [1].

COROLLARY (18,2a). *If all the zeros of an nth degree polynomial $f(z)$ lie in a convex region A, then all the zeros of the polynomial*

$$f_1(z) = f(z) - \lambda_1 f'(z) \tag{18,7}$$

lie in the convex region A_1 swept out on translating A in the magnitude and direction of the vector $n\lambda_1$. That is, $A_1 = \sum_{\mu_1} T(A, \mu_1 n\lambda_1)$, $0 \leqq \mu_1 \leqq 1$.

We have stated Cor. (18,2a) because, though weaker than Cor. (18,1), it is better suited than Cor. (18,1) to iteration. Let us define the sequence of polynomials

$$f_k(z) = f_{k-1}(z) - \lambda_k f'_{k-1}(z), \qquad k = 1, 2, \cdots, p,$$

with $f_0(z) = f(z)$. Let us also define the sequence of regions

$$A_0 = A, \qquad A_k = \sum T(A_{k-1}, n\lambda_k\mu_k), \qquad 0 \leqq \mu_k \leqq 1.$$

Clearly, $A_k = \sum T(A, n(\mu_1\lambda_1 + \mu_2\lambda_2 + \cdots + \mu_k\lambda_k))$, the sum being taken for $0 \leqq \mu_j \leqq 1$, $j = 1, 2, \cdots, p$. Fig. (18,1) illustrates the case $k = 2$ when A is a circle. By Cor. (18,2a), region A_k contains all the zeros of $f_k(z)$ if the region A_{k-1} contains all the zeros of $f_{k-1}(z)$. But, as we may write symbolically,

$$f_k(z) = (1 - D\lambda_k)f_{k-1}(z), \qquad D = d/dz,$$

we may write

$$f_p(z) = (1 - D\lambda_p)(1 - D\lambda_{p-1}) \cdots (1 - D\lambda_1)f(z).$$

This establishes the following result due to Takagi [1].

Corollary (18,2b). *Let $f(z)$ be an nth degree polynomial having all its zeros in a convex region A and let $\Lambda(z)$ be the polynomial*

$$\Lambda(z) = (1 - \lambda_1 z)(1 - \lambda_2 z) \cdots (1 - \lambda_p z).$$

Then all the zeros of the polynomial

$$F(z) = \Lambda(D)f(z), \qquad D = d/dz, \tag{18,8}$$

lie in the above-defined convex region A_p.

Of special interest is the case that $f(z) = z^n$ and that $\Lambda(z)$ is an nth degree polynomial for which the points λ_k lie in a convex sector S with vertex at the origin. Since each point $n\lambda_k$ also lies in S, each region A_k will lie in S provided the preceding region A_{k-1} lies in S. Now, the region A may be taken as the point $z = 0$. Since the corresponding region A_1 will be the line-segment joining the points $z = 0$ and $z = n\lambda_1$, the region A_1 lies in S and hence all the subsequent regions $A_2, A_3, \cdots, A_p$ lie in S.

Since the λ_k are the zeros of the polynomial

$$g(z) = z^n\Lambda(1/z)$$

we have proved

Corollary (18,2c). *If all the zeros of the polynomial*

$$g(z) = b_0 + b_1 z + \cdots + b_n z^n$$

lie in a convex sector S with the vertex at the origin, then so do also all the zeros of the polynomial

$$G(z) = b_0 + b_1z + (b_2z^2/2!) + \cdots + (b_nz^n/n!).$$

Corollary (18,2c) as stated is due to Takagi [1], but, in the special case that S is the positive or negative axis of reals, it had been previously proved by Laguerre [1, p. 31].

Thus far, we have considered linear combinations of a single polynomial and its derivative. Let us now study the linear combinations of the products $[f_1^{(j)}(z)f_2^{(n-j)}(z)]$ of the derivatives of two given polynomials $f_1(z)$ and $f_2(z)$. The first result which we shall prove is the following one due to Walsh [6]:

THEOREM (18,3). *Let the zeros of a polynomial $f_1(z)$ of degree m_1 have as locus the closed interior of a circle C_1 of center α_1 and radius r_1 and let the zeros of a polynomial $f_2(z)$ of degree m_2 have as locus the closed interior of a circle C_2 of center α_2 and radius r_2. Let the polynomial $g(z)$ be defined by the equation*

$$g(z) = \sum_{j=0}^{n} C(m_1, j)C(m_2, n-j)B_jz^{n-j} = bz^q \prod_{j=1}^{p} (z - \beta_j)$$

where the binomial coefficient $C(k, j) = 0$ if $j > k$ or $j < 0$, *where* $p + q \leqq n < m_1 + m_2$, *and where* $\beta_j \neq 0$, $\beta_j \neq 1$ for $j = 1, 2, \cdots, p$. *Then the zeros of the polynomial*

$$h(z) = \sum_{j=0}^{n} C(n, j)B_jf_1^{(j)}(z)f_2^{(n-j)}(z)$$

have as locus the point set Γ consisting of the closed interior of C_1 if $m_1 > n$, the closed interior of C_2 if $m_2 > n$ and the closed interiors of the p circles Γ_j of center γ_j and radius ρ_j, where

$$\gamma_j = \frac{\alpha_1 - \beta_j\alpha_2}{1 - \beta_j}, \qquad \rho_j = \frac{r_1 + |\beta_j| r_2}{|1 - \beta_j|}, \qquad j = 1, 2, \cdots, p.$$

To establish that Γ is the locus of the zeros of $h(z)$, we must show first that every zero of $h(z)$ lies in Γ and, secondly, that every point of Γ is a possible zero of $h(z)$. Let Z be any zero of $h(z)$. By Th. (15,4) the equation $h(Z) = 0$ being linear and symmetric in the zeros of both $f_1(z)$ and of $f_2(z)$ may be replaced by an equation obtained by coalescing all the zeros of $f_1(z)$ at a point ζ_1 in circle C_1 and coalescing all the zeros of $f_2(z)$ at a point ζ_2 in circle C_2. That is,

$$\sum_{j=0}^{n} C(n, j)C(m_1, j)C(m_2, n-j)j!(n-j)!B_j(Z - \zeta_1)^{m_1-j}(Z - \zeta_2)^{m_2-n+j}$$

$$= n!(Z - \zeta_1)^{m_1-n}(Z - \zeta_2)^{m_2}g[(Z - \zeta_1)/(Z - \zeta_2)] = 0.$$

The possible values of Z are therefore

$$Z = \zeta_1 \text{ if } m_1 > n, \qquad Z = \zeta_2 \text{ if } m_2 > n$$

and

$$Z = (\zeta_1 - \zeta_2\beta_k)/(1 - \beta_k). \tag{18,9}$$

In the first case Z is a point in or on C_1 and in the second case Z is a point in or on C_2. In the third case Z is a point in or on the circle Γ_k, as may be determined by use of Lemma (17,2a).

Conversely, if Z is any point of Γ, it is a possible zero of $h(z)$. For, we may take $f_1(z) = (z - \zeta_1)^{m_1}$ and $f_2(z) = (z - \zeta_2)^{m_2}$, choosing ζ_1 and ζ_2 as follows. If $m_1 > n$ and if Z lies in C_1, we choose $\zeta_1 = Z$ and ζ_2 as an arbitrary point in C_2. Similarly, if $m_2 > n$ and if Z lies in C_2, we choose $\zeta_2 = Z$ and ζ_1 as an arbitrary point in C_1. If, however, Z lies in Γ_k, we may according to Lemma (17,2a) so choose ζ_1 in C_1 and ζ_2 in C_2 that eq. (18,9) is satisfied.

Thus we have completely established that the point-set Γ is the locus of the zeros of $h(z)$.

As an application of Th. (18,3), let us prove

COROLLARY (18,3). *If all the zeros of an nth degree polynomial $f(z)$ lie in the circle $C : |z| \leq r$ and if all the zeros of the polynomial*

$$\phi(z) = \lambda_0 + C(m, 1)\lambda_1 z + \cdots + C(m, n)\lambda_n z^n \tag{18,10}$$

lie in the circular region K:

$$|z| \leq s|z - \tau|, \qquad s > 0, \tag{18,11}$$

then all the zeros of the polynomial

$$\psi(z) = \lambda_0 f(z) + \lambda_1 f'(z)[(\tau z)/1!] + \cdots + \lambda_n f^{(n)}(z)[(\tau z)^n/n!]$$

lie in the circle Γ:

$$|z| \leq r \max(1, s). \tag{18,12}$$

The polynomial $\psi(z)$ is of the form of the $h(z)$ given in Th. (18,3) with

$$f_1(z) \equiv f(z), \qquad f_2(z) \equiv (\tau z)^n, \qquad B_k = \lambda_k \tau^k/n!\tau^n C(n, k)$$

and consequently the corresponding $g(z)$ is

$$g(z) = (z/\tau)^n \phi(\tau/z)/n!.$$

If $\xi_1, \xi_2, \cdots, \xi_n$ are the zeros of $\phi(z)$, the zeros of $g(z)$ are $\beta_k = \tau/\xi_k$. Here circle C_1 is the same as circle C but circle C_2 is merely the point $z = 0$. According to Th. (18,3), the zeros of $\psi(z)$, not in C, lie in the circles Γ_k centers at $z = 0$ and radii

$$\rho_k = r/|1 - \beta_k| = r|\xi_k/(\xi_k - \tau)|.$$

From condition (18,11) on the zeros of $\phi(z)$ it now follows that $|\rho_k| \leq rs$. The zeros of $\psi(z)$, including those in C, therefore satisfy condition (18,12).

EXERCISES. Prove the following.

1. In Th. (18,1) if all the zeros of $f(z)$ lie in the circle $|z| \leq r_1$ and all the

zeros of $g(z)$ lie in the circle $|z| \leqq r_2$, then all the zeros of $h(z)$ lie in the circle $|z| \leqq r_1 + r_2$ [Kakeya 3].

2. In Cor. (18,2c) if all the zeros of polynomial $g(z)$ are real then all the zeros of $G(z)$ are also real [Laguerre 1, p. 31].

3. In Cor. (18,3) $f(z)$ is apolar to the polynomial $(Z - z)^m\phi[Z\tau/(Z - z)]$ if $\psi(Z) = 0$. Hence, if all the zeros of $f(z)$ lie in a convex region A and all the zeros of $\phi(z)$ lie in a region B whose inverse in the circle $|z| = 1$ is convex, then every zero Z of $\psi(z)$ has the form $Z = \alpha\beta/(\beta - \tau)$ where α is a point of A and β is a point of B. If $m > n$, assume B contains $z = \infty$.

4. In ex. (18,3) if all the zeros of $f(z)$ lie in the sector $A : \gamma \leqq \arg z \leqq \gamma + \omega \leqq \gamma + \pi$ and if all the zeros of $\phi(z)$ lie in the lune $B : \lambda \leqq \arg z/(z - \tau) \leqq \mu$ with $\mu - \lambda \leqq \pi$, then all the zeros of $\psi(z)$ lie in the sector $\gamma + \lambda \leqq \arg z \leqq \gamma + \omega + \mu$. If $m > n$, assume $\mu\lambda \leqq 0$.

5. If the zeros of $f(z) = e^{az}P(z)$, where $P(z)$ is an nth degree polynomial and a is a constant, lie in a circular region C, the zeros of $f'(z)$ lie in region C and in the region C' obtained on translating C in the magnitude and direction of the vector $(-na^{-1})$. Hint: Use Cor. (18,1).

6. If the pth degree $P(z)$ and the qth degree polynomial $Q(z)$ have all their zeros in the same circular region C, then the zeros of the derivative of $f(z) = e^{P(z)}Q(z)$ lie in region C and the p circular regions C_j' obtained on translating C in the magnitude and direction of the vectors $[\omega_j(-q/ap)^{1/p}]$ $(j = 1, 2, \cdots, p)$ where ω_j are the pth roots of unity and a is the coefficient of the pth degree term in $P(z)$ [Walsh 6].

7. If all the zeros of the kth degree polynomial $f_1(z)$ lie inside a circle K and all those of the rth degree polynomial $f_2(z)$, $r < k$, lie outside K, then inside K lie all the zeros of the polynomial

$$h(z) = \sum(-1)^j C(k - r + j, j) f_1^{(j)}(z) f_2^{(r-j)}(z), \qquad 0 \leqq j \leqq r \qquad \text{[Curtiss 1].}$$

8. Let $F(z)$ and $G(z)$ be polynomials which have all their zeros in the strip $|\Im(z)| \leqq a$, $a > 0$. Then all the zeros of $H(z) = \sum_0^\infty (t^k/k!)F^{(k)}(z)G^{(k)}(z)$, $t < 0$, also lie in this strip [De Bruijn 3]. Hint: Let

$$f(z) = \sum_0^\infty z^k F^{(k)}(w)/k! = F(z + w) \text{ and } g(z) = \sum_0^\infty z^k G^{(k)}(w)/k! = G(z + w)$$

and apply ex. (16,6) taking $h(z)$ as $h_1(z)$.

CHAPTER V

THE CRITICAL POINTS OF A RATIONAL FUNCTION WHICH HAS ITS ZEROS AND POLES IN PRESCRIBED CIRCULAR REGIONS

19. A two-circle theorem for polynomials. The Lucas Theorem which we developed in sec. 6 states that any convex region K enclosing all the zeros of a polynomial $f(z)$ contains also all the critical points of $f(z)$. Furthermore, as we remarked in sec. 6, every point interior to or on the boundary of K is the critical point of at least one polynomial which has all its zeros in K.

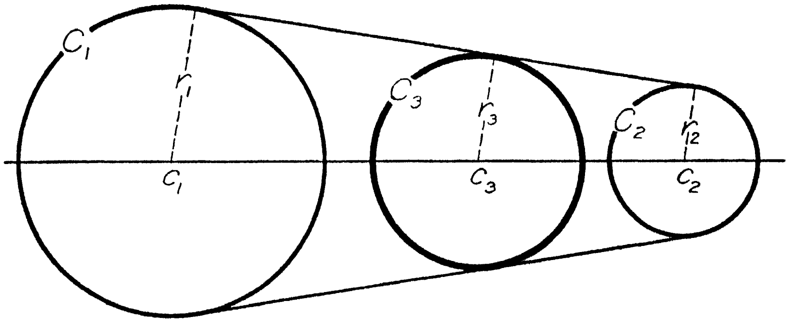

FIG. (19, 1)

Let us now consider the class T of all polynomials $f(z)$ of degree n which have n_1 zeros in or on a circle C_1, n_2 zeros in or on a circle C_2, etc., and n_p zeros in or on a circle C_p, with $n_1 + n_2 + \cdots + n_p = n$. If K denotes the smallest convex region enclosing all the circles C_j $(j = 1, 2, \cdots, p)$, all the critical points of every $f(z)$ in T will lie in K, but not every point of K will necessarily be a critical point of some $f(z)$ in T. Let us now determine the precise locus of the critical points of the polynomials of class T.

We begin with the case $p = 2$ which was first studied by Walsh [2]. We shall state his result as

WALSH'S TWO-CIRCLE THEOREM (Th. 19,1). *If the locus of the zeros of the n_1-degree polynomial $f_1(z)$ is the closed interior of the circle C_1 with center c_1 and radius r_1 and the locus of the zeros of the n_2-degree polynomial $f_2(z)$ is the closed interior of the circle C_2 with center c_2 and radius r_2, then the locus of the zeros of the derivative of the product $f(z) = f_1(z)f_2(z)$ consists of the closed interiors of C_1 if $n_1 > 1$, of C_2 if $n_2 > 1$ and of a third circle C with center c and radius r where*

$$c = \frac{n_1c_2 + n_2c_1}{n_1 + n_2}, \qquad r = \frac{n_1r_2 + n_2r_1}{n_1 + n_2}. \tag{19,1}$$

In a sense, the third circle C is the weighted average of the two given circles C_1 and C_2. (In fig. (19,1) $C = C_3$, $r = r_3$ and $c = c_3$.) It has with C_1 and C_2 a common center of similitude and its center is the centroid of the system of two particles, one of mass n_2 at c_1 and the other of mass n_1 at c_2.

To prove Th. (19,1), let us note that, if Z is any zero of $f'(z)$,

$$0 = f'(Z) = f_1'(Z)f_2(Z) + f_1(Z)f_2'(Z). \tag{19,2}$$

This is an equation which is linear and symmetric in the zeros of $f_1(z)$ and in the zeros of $f_2(z)$. By Th. (15,4), Z will also satisfy the equation obtained by substituting into eq. (19,2)

$$f_1(z) = (z - \zeta_1)^{n_1}, \qquad f_2(z) = (z - \zeta_2)^{n_2}$$

where ζ_1 and ζ_2 are suitably chosen points, the first in C_1 and the second in C_2. That is, Z satisfies the equation

$$n_1(Z - \zeta_1)^{n_1-1}(Z - \zeta_2)^{n_2} + n_2(Z - \zeta_1)^{n_1}(Z - \zeta_2)^{n_2-1} = 0$$

and thus has the values of

$$Z = \zeta_1, \text{ if } n_1 > 1; \qquad Z = \zeta_2, \text{ if } n_2 > 1;$$
$$Z = (n_2\zeta_1 + z_2\zeta_2)/(n_2 + n_1).$$

Obviously the first Z is a point in C_1 and the second Z is a point in C_2. That the third Z is a point in C may be verified by setting $p = 2$, $m_1 = n_2/(n_1 + n_2)$ and $m_2 = n_1/(n_1 + n_2)$ in Lemma (17,2a). Thus we have proved that every zero Z of $f'(z)$ lies in at least one of the circles C_1, C_2 and C.

Conversely, as in the proof of Th. (18,3), we may show that any point Z in or on the circle C_1, C_2 or C is a zero of the derivative of $f(z) = f_1(z)f_2(z)$ for suitably chosen polynomials $f_1(z)$ and $f_2(z)$ having all their zeros in C_1 and C_2 respectively. We may thereby complete the proof of Th. (19,1).

Concerning the number of zeros of $f'(z)$, we may as in Walsh [2] deduce the following result.

Corollary (19,1). *If the closed interiors of the circles C_1, C_2 and C of Th. (19,1) have no point in common, the number of zeros of $f'(z)$ which they contain is respectively $n_1 - 1$, $n_2 - 1$ and 1.*

For, if ξ_1 is any point in C_1 and ξ_2 any point in C_2, then we may allow all the n_1 zeros of $f(z)$ in C_1 to approach ξ_1 along regular paths entirely in C_1 and similarly allow all the n_2 zeros of $f(z)$ in C_2 to approach ξ_2 along regular paths in C_2. Thus ξ_1 and ξ_2 become zeros of $f'(z)$ of the respective multiplicities $n_1 - 1$ and $n_2 - 1$, the remaining zero of $f'(z)$ then being a point of C. During this process, no zero of $f'(z)$ can enter or leave C_1, C_2 or C. Hence, the number of zeros in C_1, C_2 and C was also originally $n_1 - 1$, $n_2 - 1$ and 1.

By a similar method we may establish the following somewhat more general result concerning the function $F(z) = \sum [m_i/(z - z_i)]$, where the m_i are arbitrary positive numbers.

THEOREM (19,2). *If all the points* z_j , $1 \leqq j \leqq p_1$, *lie in or on the circle* C_1 *and if all the points* z_j , $p_1 + 1 \leqq j \leqq p_1 + p_2$, *lie in or on a circle* C_2 , *then any zero of the function*

$$F(z) = \sum_{j=1}^{p_1+p_2} \frac{m_j}{z - z_j}, \qquad m_j > 0, \text{ all } j,$$

if not in or on C_1 *or* C_2 , *lies in the circle* C *defined in Th.* (19,1) *with*

$$n_1 = \sum_{j=1}^{p_1} m_j \quad \text{and} \quad n_2 = \sum_{j=p_1+1}^{p_1+p_2} m_j .$$

As an application of Th. (19,2), we shall derive the following Mean-Value Theorem for polynomials.

THEOREM (19,3). *Let the circle* C_1 *with center* c_1 *and radius* r_1 *enclose all the points in which a pth degree polynomial* $P(z)$ *assumes the value* A *and let the circle* C_2 *with center* c_2 *and radius* r_2 *enclose all the points in which* $P(z)$ *assumes the value* B. *Then if* n_1 *and* n_2 *are arbitrary positive numbers, the circles* C_1 *and* C_2 *and a third circle* C *with center* $c = (n_1c_1 + n_2c_2)/(n_1 + n_2)$ *and radius* $r = (n_1r_1 + n_2r_2)/(n_1 + n_2)$ *contain all the points at which* $P(z)$ *assumes the average value* $M = (n_1A + n_2B)/(n_1 + n_2)$.

This theorem is stated and proved in Pólya-Szegö [1, vol. 2, p. 61] in the case that n_1 and n_2 are positive integers. To prove it in the more general case, let us denote by z_j , $1 \leqq j \leqq p$, the points where $P(z) = B$ and by z_j , $p + 1 \leqq j \leqq 2p$, the points where $P(z) = A$. Thus,

$$(19,3) \qquad P(z) - B = \prod_{j=1}^{p} (z - z_j), \qquad P(z) - A = \prod_{j=p+1}^{2p} (z - z_j).$$

If Z denotes any point at which $P(z) = M$, then

$$(n_1 + n_2)[P(Z) - M] = n_1[P(Z) - A] + n_2[P(Z) - B] = 0.$$

Hence,

$$(19,4) \qquad \frac{n_1P'(Z)}{P(Z) - B} + \frac{n_2P'(Z)}{P(Z) - A} = 0.$$

Substituting from eq. (19,3) into eq. (19,4), we find

$$\sum_{j=1}^{p} \frac{n_1}{Z - z_j} + \sum_{j=p+1}^{2p} \frac{n_2}{Z - z_j} = 0.$$

According to Th. (19,2), therefore, point Z must lie in at least one of the circles C_1 , C_2 and C.

EXERCISES. Prove the following.

1. Th. (6,2) is the special case of Th. (19,1) in which $C_1 = C_2$.
2. Let $m_1^{(j)} : m_2^{(j)}$ ($j = 1, 2, \cdots , q$) denote the ratios in which the line-segment (c_1 , c_2) is divided by the q distinct zeros of the kth derivative of the

$g(z) = (z - c_1)^{n_1}(z - c_2)^{n_2}$. Let K_j denote the circle with center at $(m_2^{(j)}c_1 + m_1^{(j)}c_2)/(m_1^{(j)} + m_2^{(j)})$ and radius $(m_2^{(j)}r_1 + m_1^{(j)}r_2)/(m_1^{(j)} + m_2^{(j)})$. Then the locus of the zeros of the kth derivative of the $f(z)$ of Th. (19,1) is composed of the circle C_1 if $k < n_1$, the circle C_2 if $k < n_2$ and the q circles K_j. If any K_j is exterior to all the other K_j, it contains a number of zeros of $f^{(k)}(z)$ equal to the multiplicity of the corresponding zero of $g^{(k)}(z)$ [Walsh 6, pp. 175-6].

3. Th. (19,1) and ex. (19,2) are special cases of Th. (18,3).

4. If every zero of an n_1-degree polynomial $f_1(z)$ lies in or on the circle $C_1 : |z| \leqq r_1$ and if every zero of an n_2-degree polynomial $f_2(z)$ lies on or outside the circle $C_2 : |z| \geqq r_2$, where $r_2 \geqq (n_2r_1/n_1)$, then every zero of the derivative of the product $f(z) = f_1(z)f_2(z)$ lies in or on C_1, on or outside C_2 or on or outside the circle C: $|z| \geqq r = (n_1r_2 - n_2r_1)/(n_1 + n_2)$. Furthermore, if $r > r_1$, exactly $n_1 - 1$ zeros of $f'(z)$ lie in C_1 and exactly n_2 lie on or outside C.

5. If an nth degree polynomial $f(z)$ has a k-fold zero at a point P and its remaining $n - k$ zeros in a circular region C, then $f'(z)$ has its zeros at P, in C and in a circular region C' formed by shrinking C towards P as center of similitude in the ratio $1 : k/n$. If C and C' have no point in common, they contain respectively $n - k - 1$ zeros and one zero of $f'(z)$ [Walsh 1b, p. 115].

6. Let $F(z)$ be an nth degree polynomial whose zeros are symmetric in the origin 0. Let 0 be a k-fold zero of $F(z)$ and let all the other zeros of $F(z)$ lie in the closed interior of an equilateral hyperbola H with center at 0. Then, except for a $(k - 1)$-fold zero at 0, all the zeros of $F'(z)$ lie in the closed interior of the equilateral hyperbola obtained by shrinking H towards 0 in the ratio $n^{1/2} : k^{1/2}$. Hint: Apply ex. (19,5) to $f(w) = f(z^2) = [F(z)]^2$, taking the circular region C as the half-plane $\Re(w) \geqq a > 0$ [Walsh 17].

20. **Two-circle theorems for rational functions.** The question raised in sec. 19 concerning the derivatives of a product of two polynomials may also be asked concerning the finite zeros of the derivative of their quotient. Here the answer, also due to Walsh [1b, p. 115], reads as follows.

THEOREM (20,1). *If the polynomial $f_1(z)$ of degree n_1 has all its zeros in or on a circle C_1 with center c_1 and radius r_1, and if the polynomial $f_2(z)$ of degree n_2 has all its zeros in or on a circle C_2 with center c_2 and radius r_2 and if $n_1 \neq n_2$, then the finite zeros of the derivative of the quotient $f(z) = f_1(z)/f_2(z)$ lie in C_1, C_2 and a third circle C with center c and radius r where*

$$c = \frac{n_2c_1 - n_1c_2}{n_2 - n_1}, \qquad r = \frac{n_2r_1 + n_1r_2}{|n_2 - n_1|}.$$

Under these hypotheses if $n_1 = n_2$ and if the closed interiors of C_1 and C_2 have no point in common, then these two circles contain all the zeros of $f'(z)$.

The proof of Th. (20,1) is similar to that of Th. (19,1) and will be left to the reader. He should, however, note that, if $n_1 = n_2$ and if the interiors of circles C_1 and C_2 did overlap, the zeros of both $f_1(z)$ and $f_2(z)$ could be made to coincide

at the common points of C_1 and C_2. The corresponding quotient $f(z)$ would then be constant and its derivative identically zero. That is, if $n_1 = n_2$ and if C_1 and C_2, originally without a common point, are allowed to expand, the locus of the zeros of $f'(z)$ changes abruptly to the entire plane when C_1 and C_2 become tangent.

Th. (20,1) is essentially a proposition concerning the finite zeros of the rational function

$$F(z) = \sum_{j=1}^{p_1+p_2} \frac{m_j}{z - z_j}$$

in which $m_j > 0$ for $1 \leqq j \leqq p_1$ and $m_j < 0$ for $p_1 + 1 \leqq j \leqq p_1 + p_2$ and in which all the z_j, $1 \leqq j \leqq p_1$ are points in or on the circle C_1 and all the z_j, $p_1 + 1 \leqq j \leqq p_1 + p_2$, are points in or on the circle C_2. The numbers n_1 and n_2 are here

$$n_1 = \sum_{j=1}^{p_1} m_j \qquad \text{and} \qquad n_2 = \sum_{j=p_1+1}^{p_1+p_2} m_j .$$

In the case $n_1 = n_2$, we are dealing with the logarithmic derivative of a function of type (10,1) in which the total "mass" is zero and therefore of which, according to sec. 10, the zeros are invariant under linear transformations. We may therefore replace the interiors of circles C_1 and C_2 by arbitrary circular regions C_1 and C_2. We may also introduce the binary forms

$$\Phi_1(\xi, \eta) = \sum_{k=0}^{n} a_k \xi^k \eta^{n-k} = \eta^n f_1(\xi/\eta),$$

$$\Phi_2(\xi, \eta) = \sum_{k=0}^{n} b_k \xi^k \eta^{n-k} = \eta^n f_2(\xi/\eta). \tag{20,1}$$

The jacobian of these forms is

$$\begin{aligned} J(\xi, \eta) &= \frac{\partial \Phi_1}{\partial \xi}\frac{\partial \Phi_2}{\partial \eta} - \frac{\partial \Phi_2}{\partial \xi}\frac{\partial \Phi_1}{\partial \eta} \\ &= n\eta^{2(n-1)}[f_1'(\xi/\eta)f_2(\xi/\eta) - f_2'(\xi/\eta)f_1(\xi/\eta)] \\ &= n\eta^{2(n-1)}f'(\xi/\eta)[f_2(\xi/\eta)]^2 \end{aligned}$$

where $f(z) = f_1(z)/f_2(z)$. Since the zeros of $J(\xi, \eta)$ are the finite zeros of $f'(\xi/\eta)$ and possibly the point at infinity, we may restate the last part of Th. (20,1) in the form due to Bôcher [4].

THEOREM (20,2). *If each zero of the form $\Phi_1(\xi, \eta)$ lies in a circular region C_1, if each zero of the form $\Phi_2(\xi, \eta)$ lies in a circular region C_1 and if the regions C_1 and C_2 have no points in common, then no finite zero of the jacobian of the two forms lies exterior to both regions C_1 and C_2.*

EXERCISES. Prove the following.

1. In Th. (20,1), if $f_2(z)$ has no multiple zeros, if $n_1 \neq n_2$, and if the circles C_1, C_2 and C have no point in common, then $f'(z)$ has in these circles respectively $n_1 - 1$, $n_2 - 1$ and 1 zero(s) [Walsh 1b, p. 115].

2. Laguerre's Theorem (Th. 13,1) is a special case of Th. (20,1).

3. Let positive particles of total mass n be placed at certain points of a circular region C_1 on the unit sphere S and negative particles of total mass $(-n)$ at certain points of a circular region C_2 on S. If the regions C_1 and C_2 have no common points, then no point on S exterior to both regions C_1 and C_2 can be a point of equilibrium in this field. Thus, obtain another proof of Th. (20,1) in the case $n_1 = n_2$ [Bôcher 4].

4. If the circle C_1 with center c_1 and radius r_1 contains all the points where a pth degree polynomial $P(z)$ assumes the value A and if the circle C_2 with center c_2 and radius r_2 contains all the points where $P(z)$ assumes the value B, then the points where $P(z)$ assumes the value $(n_1A - n_2B)/(n_1 - n_2)$, $n_1 > n_2 > 0$, lie in C_1, C_2, and the circle C of Th. (20,1).

5. Let $f(z)$ be a polynomial of degree m and $g(z)$ a polynomial of degree $n \neq m$. Let all the zeros of $g(z)$ lie in circular region R bounded by a circle C. Let w be any point on C and let ζ be defined by the equation

$$\frac{\lambda f'(w)}{f(w)} - \frac{g'(w)}{g(w)} = \frac{\lambda m - n}{w - \zeta}.$$

Then, if ζ also lies on C_2 and if $\lambda > 0$ and $\lambda \neq [n/m]$, at least one zero of $f(z)$ lies in R. Hint: Apply Th. (15,4) and Lemma (17,2a) [Obrechkoff 4].

21. **The general case.** In generalization of secs. 19 and 20, we shall now study the derivative of a rational function which has its zeros and poles distributed over any number of prescribed circular regions. The results which we shall obtain are due to Marden [3] and [10]. We begin with

THEOREM (21,1). *For $j = 0, 1, \cdots, p$ let $f_j(z)$ denote a polynomial of degree n_j having all of its zeros in the circular region $\sigma_j C_j(z) \leqq 0$ where $\sigma_j = \pm 1$ and*

$$C_j(z) = |z - c_j|^2 - r_j^2. \tag{21,1}$$

Then every finite zero Z of the derivative of the rational function

$$f(z) = \frac{f_0(z)f_1(z) \cdots f_q(z)}{f_{q+1}(z)f_{q+2}(z) \cdots f_p(z)}, \qquad 0 \leqq q \leqq p, \tag{21,2}$$

satisfies at least one of the $p + 2$ inequalities

$$\sigma_j C_j(Z) \leqq 0, \qquad j = 0, 1, \cdots, p,$$

(21,3)

$$\frac{E(Z)}{C_0(Z)C_1(Z) \cdots C_p(Z)} = \sum_{j=0}^{p} \frac{nm_j}{C_j(Z)} - \sum_{j=0, k=j+1}^{p} \frac{m_j m_k \tau_{jk}}{C_j(Z)C_k(Z)} \leqq 0$$

where $m_j = n_j$ *or* $-n_j$ *according as* $j \leqq q$ *or* $j > q$, $n = \sum_0^p m_j$,

$$\tau_{jk} = | c_j - c_k |^2 - (\mu_j r_j - \mu_k r_k)^2, \qquad \mu_j = \sigma_j \operatorname{sg} m_j \,.$$

Before taking up its proof, let us interpret Th. (21,1) from a geometric standpoint. According as $\sigma_j = 1$ or -1, the region C_j defined by the inequality $\sigma_j C_j(z) \leqq 0$ is the closed interior or the closed exterior of the circle with c_j as

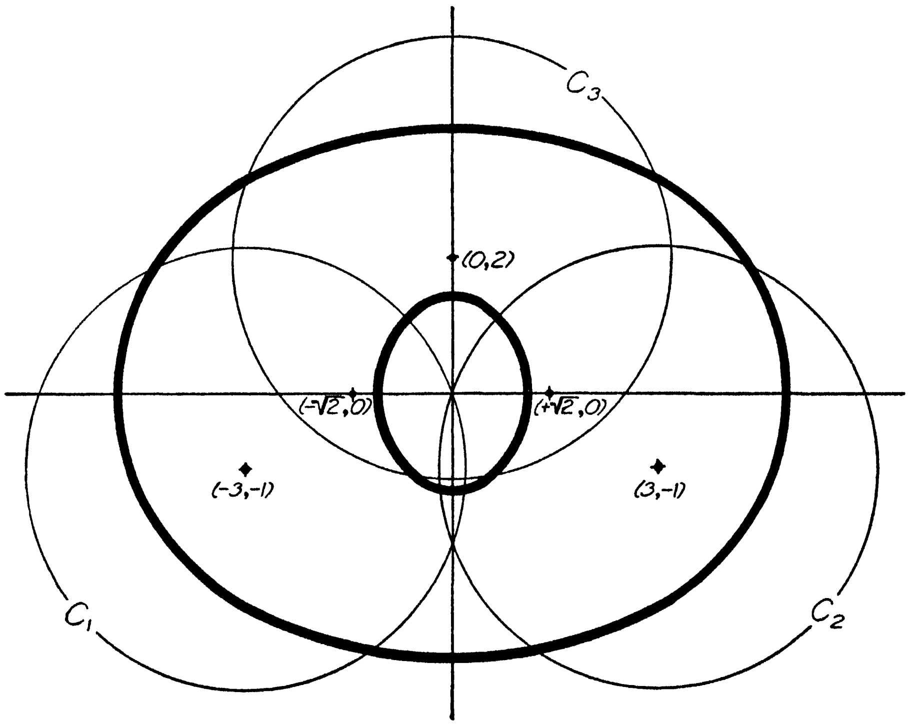

FIG. (21, 1)

center and r_j as radius. According as $\mu_j\mu_k = 1$ or -1, the quantity τ_{jk} when positive is the square of the common external or internal tangent of the circles $C_j(z) = 0$ and $C_k(z) = 0$. When $n \neq 0$, the equation $E(x + iy) = 0$ may be written in the form

$$(x^2 + y^2)^p + \phi(x, y) = 0$$

where $\phi(x, y)$ is a real polynomial with a combined degree of at most $2p - 1$ in x and y. Being of this form, $E(z) = 0$ represents a so-called p-circular $2p$-ic curve, a curve of degree $2p$ which passes p times through each circular point at infinity. As such, the curve $E(z) = 0$ consists of at most p branches, each of which is a finite closed Jordan curve. In short, when $n \neq 0$, Th. (21,1) implies that each zero Z of $f'(z)$ lies in at least one of the given circular regions C_j or lies in one of the regions bounded by the p-circular $2p$-ic curve $E(z) = 0$.

For example, when $p = 2$, equation $E(z) = 0$ becomes

$$
\begin{aligned}
&nm_0C_1(z)C_2(z) + nm_1C_2(z)C_0(z) + nm_2C_0(z)C_1(z) \\
(21,4) \qquad & \qquad -m_0m_1\tau_{01}C_2(z) - m_1m_2\tau_{12}C_0(z) - m_2m_0\tau_{20}C_1(z) = 0.
\end{aligned}
$$

For $n \neq 0$, (21,4) is the equation of a bicircular quartic, a result which for the m_j positive integers and the C_j interiors of circles coincides with the result due to Walsh [5]. Fig. (21,1) illustrates the case that $m_1 = m_2 = m_3$ and that regions C_1, C_2 and C_3 are the interiors of circles of radii $r_1 = r_2 = r_3 = 10^{1/2}$ with centers at the points $c_1 = -3 - i$, $c_2 = 3 - i$ and $c_3 = 2i$. In that case, curve $E(z) = 0$ consists of two nested ovals.

When $n = 0$, we may give a similar interpretation of Th. (21,1). The equation $E(z) = 0$ then represents in general a $(p-1)$-circular $2(p-1)$-ic curve. In this case, we must take the precaution that not all the regions C_j have a point in common. For, if t were such a point, we could reduce $f(z)$ to a constant by concentrating at t all the zeros of all the $f_j(z)$, whereupon $f'(z) \equiv 0$ making every point in the plane a possible position of Z. In other words, if we wish a nontrivial result in the case $n = 0$, we must assume that no point is common to all the regions C_j.

Proceeding now to the proof of Th. (21,1), let us denote the numerator in eq. (21,3) by $F_1(z)$ and the denominator by $F_2(z)$. For any zero Z of $f'(z)$, the expression $F(Z) = F_2(Z)F_1'(Z) - F_1(Z)F_2'(Z) = 0$ is one which is linear and symmetric in the zeros of each $f_j(z)$. By Th. (15,4), we may select a point ζ_j in each region C_j such that Z will also satisfy the equation obtained from $F(Z) = 0$ by setting $f_j(z) = (z - \zeta_j)^{n_j}$. Thus we find that either $Z = \zeta_j$ for some value of j for which $n_j > 1$ or Z satisfies the equation

$$
(21,5) \qquad \sum_{j=0}^{p} [m_j/(Z - \zeta_j)] = 0.
$$

In the first case Z lies in a region C_j and thus satisfies the jth of the inequalities (21,3). In the second case Z lies in the locus R described by the roots of equation (21,5) when $\zeta_0, \zeta_1, \cdots, \zeta_p$ are allowed to vary independently over the circular regions $C_0, C_1, \cdots, C_p$ respectively.

For the purpose of determining R, let us establish

LEMMA (21,1). *If the points ζ_j lie in the circular regions C_j $(j = 0, 1, \cdots, p)$, then every root Z of eq. (21,5) lies in a region C_j or satisfies the inequality*

$$
(21,6) \qquad \left| \sum_{j=0}^{p} \frac{m_j(c_j - z)}{C_j(z)} \right|^2 - \left(\sum_{j=0}^{p} \frac{| m_j | r_j}{| C_j(z) |} \right)^2 \leqq 0.
$$

Let us first choose Z as any fixed point which lies exterior to all the regions C_j. Then by Lemma (12,1) the point $w_j = (\bar{Z} - \bar{\zeta}_j)^{-1}$ lies inside the circle Γ_j with center γ_j and radius ρ_j where

$$
\gamma_j = (Z - c_j)/C(Z), \qquad \rho_j = r_j/| C_j(Z) |.
$$

Hence, by Lemma (17,2a), the point $w = \sum_0^p m_j w_j$ lies in the circle Γ with center γ and radius ρ where

$$(21,7) \qquad \gamma = \sum_{j=0}^{p} m_j \gamma_j = \sum_{j=0}^{p} [m_j(Z - c_j)/C_j(Z)],$$

$$(21,8) \qquad \rho = \sum_{j=0}^{p} | m_j | \rho_j = \sum_{j=0}^{p} [| m_j | r_j / | C_j(Z) |].$$

That is,

$$(21,9) \qquad | w - \gamma |^2 - \rho^2 \leqq 0.$$

Now, let us specialize Z to be a root of eq. (21,5). Then, as the points ζ_j vary over the regions C_j, point w assumes the value of zero at least once. That is, $w = 0$ must satisfy ineq. (21,9); viz.,

$$(21,10) \qquad | \gamma |^2 - \rho^2 \leqq 0.$$

On substituting from eqs. (21,7) and (21,8) into (21,10), we finally obtain ineq. (21,6).

To complete the proof of Th. (21,1), we need now to show that the left sides of ineqs. (21,3) and (21,6) are identical. Using the identity

$$(Z - c_j)(\bar{Z} - \bar{c}_k) + (\bar{Z} - \bar{c}_j)(Z - c_k)$$

$$= | Z - c_j |^2 + | Z - c_k |^2 - | c_j - c_k |^2,$$

we find from (21,7) that

$$| \gamma |^2 = \sum_{j=0}^{p} \frac{m_j^2 \, | Z - c_j |^2}{C_j(Z)^2}$$

$$+ \sum_{j=0, k=j+1}^{p} \frac{m_j m_k(| Z - c_j |^2 + | Z - c_k |^2 - | c_j - c_k |^2)}{C_j(Z) C_k(Z)}.$$

Using the notation of Th. (21,1) and the hypothesis that $\sigma_j C_j(Z) > 0$ and thus $| C_j(Z) | = C_j(Z)/\sigma_j$, we find from (21,8) that

$$\rho^2 = \sum_{j=0}^{p} \frac{m_j^2 r_j^2}{C_j(Z)^2} + \sum_{j=0, k=j+1}^{p} \frac{2 m_j m_k \mu_j \mu_k r_j r_k}{C_j(Z) C_k(Z)}.$$

Finally, using eq. (21,1), we find that

$$| \gamma |^2 - \rho^2 = \sum_{j=0}^{p} \frac{m_j^2}{C_j(Z)} + \sum_{j=0, k=j+1}^{p} \frac{m_j m_k [C_j(Z) + C_k(Z) - \tau_{jk}]}{C_j(Z) C_k(Z)}$$

which reduces at once to the expression in ineq. (21,3) for $E(z)$.

We shall now establish the following converse of Th. (21,1).

Theorem (21,2). *Let Z be any point which satisfies the inequality*

(21,11) $$\sigma_0\sigma_1 \cdots \sigma_p E(Z) \leqq 0.$$

Then Z is a zero of the derivative of a function of type (21,2) *with each $f_j(z) = (z - \zeta_j)^{n_j}$ and with ζ_j a suitably chosen point in the region $\sigma_j C_j(Z) \leqq 0$.*

First, let us suppose that Z lies exterior to all the regions C_j, i.e., that $\sigma_j C_j(Z) > 0$ for all j. Then ineq. (21,11) is identical with ineq. (21,6). We also note that Lemma (17,2a) concerns the *locus* of point w and that in Lemma (12,1) the *locus* of point w_1, as Z remains fixed and as z_1 varies over the interior or exterior of circle C, is the interior or exterior of the circle C'. In this case, therefore, we may, without difficulty, retrace the steps which lead to Lemma (21,1) and thus prove Th. (21,2).

Secondly, let us suppose that Z lies interior to one region, say C_0, and exterior to the remaining C_j. Using Lemma (12,1) and the notation employed in the proof of Lemma (21,1), we find that w_0 lies outside the circle Γ_0 and the remaining w_j lie inside the circles Γ_j. The locus of point $w = \sum_{j=0}^{p} m_j w_j$ is then the region

(21,12) $$| w - \gamma |^2 - \rho^2 \geqq 0$$

where

$$\gamma = \sum_{j=1}^{p} m_j\gamma_j = \sum_{j=1}^{p} [m_j(Z - c_j)/C_j(Z)],$$

$$\rho = | m_0 | \rho_0 - \sum_{j=1}^{p} | m_j | \rho_j = (| m_0 | r_0/| C_0(Z) |) - \sum_{j=1}^{p} (| m_j | r_j/| C_j(Z) |),$$

provided $\rho > 0$. Since here

(21,13) $$\sigma_0 C_0(Z) < 0; \qquad \sigma_j C_j(Z) > 0 \qquad \text{for } j = 1, 2, \cdots, p,$$

we may write

$$\rho = - \sum_{j=0}^{p} [m_j\mu_j r_j/C_j(Z)].$$

If we insert these values of γ and ρ and also $w = 0$ into ineq. (21,12), we find (21,12) becomes

$$E(Z)/C_0(Z)C_1(Z) \cdots C_p(Z) \geqq 0$$

which, because of (21,13), reduces to (21,11). Hence, if $\rho > 0$, ineq. (21,11) implies that point $w = 0$ satisfies (21,12) and therefore that points ζ_j may be chosen in the regions C_j making Z a root of eq. (21,5). If $\rho \leqq 0$, the locus of w is the entire plane; the point $w = 0$ is surely a point of the locus and Z is a root of eq. (21,5) for a suitable choice of points ζ_j.

Finally, if Z lies in two or more regions C_j, the locus of point w is the entire plane and again the point Z will be a root of eq. (21,5) for suitably selected ζ_j.

Thus, we have completed the proof of Theorem (21,2).

Theorems (21,1) and (21,2) do not in all cases completely specify R, the locus of the roots of eq. (21,5) when the ζ_j have the circular regions C_j as their respective loci. For example, in the case that the bicircular quartic (21,4) consists of two nested ovals, the requirement (21,11) of Th. (21,2) merely ensures that the region between the ovals belongs to R.

It is clear, however, from the proof of Th. (21,2) that the inequality opposite to (21,11), namely $\sigma_0\sigma_1 \cdots \sigma_p E(Z) > 0$, may be satisfied only under one of the following two circumstances. Either the point Z lies in just one region C_j and simultaneously

$$\sum_{j=0}^{p} [m_j \mu_j r_j / C_j(Z)] \geqq 0, \tag{21,14}$$

or it lies at a point common to at least two regions C_j.

That the locus R may in fact possess a component which is not a simply-connected region is illustrated by the following example suggested by Professor Walsh. Let us take $m_j = 1$ for $j = 0, 1, \cdots, p$. Let us choose the region C_0 as merely the origin and each region C_j, $j = 1, 2, \cdots, p$, as the circle with a center at the point $z = e^{2\pi i j/p}$ and with a radius r such that $\sin(\pi/p) < r < 1$. Each circle C_j, $j = 1, 2, \cdots, p$, obviously overlaps its two neighboring circles C_j but does not contain the origin. Being but a simple zero of $f(z)$, the origin cannot be a zero of $f'(z)$ no matter what the positions of the remaining zeros of $f(z)$ may be within the regions C_j, $j = 1, 2, \cdots, p$. Clearly, therefore, the locus R completely surrounds the origin but does not include it. Thus, R consists of at least one region which is not simply-connected.

Exercises. Prove the following.

1. If $m_1 = m_2 = m_3$, if $c_1 = -3 - i$, $c_2 = 3 - i$ and $c_3 = 2i$, and if $r_1 = r_2 = r_3 = r$, then the bicircular quartic (21,4) consists of (a) two ovals, neither enclosing the other, if $r < 3^{1/2} - 1$; (b) a single oval if $3^{1/2} - 1 < r < 3^{1/2} + 1$; (c) two ovals one enclosing the other, if $r > 3^{1/2} + 1$.

2. If Z is taken as a root of the equation $\lambda + \sum_1^p [m_j/(Z - \zeta_j)] = 0$, then (21,6) must be replaced by the inequality

$$\left| \lambda + \sum_{j=1}^{p} \frac{m_j(\bar{c}_j - \bar{Z})}{C_j(Z)} \right|^2 - \left(\sum_{j=1}^{p} \frac{| m_j | \sigma_j r_j}{C_j(Z)} \right)^2 \leqq 0. \tag{21,15}$$

Hint: $w = \lambda$ must satisfy eq. (21,9).

3. If the hypotheses of Th. (21,1) are satisfied, and if $F(z) = f(z)/f_0(z)$, then each zero of the linear combination $F'(z) + \lambda F(z)$ lies in one of the regions $C_1, C_2, \cdots, C_p$ or in the point set S bounded by the branches of p-circular $2p$-ic curve

$$\sum_{j=1}^{p} \frac{| \lambda |^2 m_j \Gamma_j(z)}{n C_j(z)} - \sum_{j=1, k=j+1}^{p} \frac{m_j m_k \tau_{jk}}{C_j(z) C_k(z)} = 0, \tag{21,16}$$

where

$$\Gamma_i(z) = |z - (c_i - n\lambda^{-1})|^2 - r_i^2 .$$

Hint: Use Th. (15,4) and ex. (21,2) [Marden 10].

4. A result similar to Th. (21,1) is valid when one or more of the regions C_i are half-planes $\sigma_i L_i(z) \leqq 0$ where $\sigma_i = \pm 1$ and where $L_i(z) = \Re(ze^{i\alpha_i}) - h_i$ with α_i and h_i real.

22. **Some important special cases.** We shall now consider under Th. (21,1) a number of special cases which involve three or more polynomials $f_i(z)$ and in which the p-circular $2p$-ic curve $E(z) = 0$ degenerates into one or more circles.

We begin with the case $p = 2$. When $n = 0$, eq. (21,4) with all subscripts increased by one reduces to the equation

$$m_2 m_3 \tau_{23} C_1(z) + m_3 m_1 \tau_{31} C_2(z) + m_1 m_2 \tau_{12} C_3(z) = 0, \tag{22,1}$$

the equation of a circle. On the other hand, for this special case eq. (21,5) becomes on replacing Z by z

$$\frac{m_1}{z - \zeta_1} + \frac{m_2}{z - \zeta_2} + \frac{-(m_1 + m_2)}{z - \zeta_3} = 0 \tag{22,2}$$

which, solved for $-m_1/m_2$, may be written as

$$\frac{(z - \zeta_2)(\zeta_3 - \zeta_1)}{(z - \zeta_1)(\zeta_3 - \zeta_2)} = -\frac{m_2}{m_1}. \tag{22,3}$$

In other words, the region bounded by circle (22,1) is the locus described by a point z which forms with ζ_1, ζ_2 and ζ_3 the constant cross-ratio (22,3), as the ζ_i describe their regions C_i.

These results may be summarized in the form of two theorems both due to Walsh [1].

Walsh's Cross-Ratio Theorem (Th. 22,1). *If the points ζ_1, ζ_2, ζ_3 varying independently have given circular regions as their loci, then any point z forming a constant cross-ratio with ζ_1, ζ_2 and ζ_3 also has a circular region as its locus.*

Theorem (22,2). *For each $j = 1, 2, 3$, let $f_j(z)$ be a polynomial of degree n_j having all its zeros in a circular region C_j. If $n_1 + n_2 = n_3$ and if no point is common to all the regions C_j, then each finite zero of the derivative of the function*

$$f(z) = f_1(z)f_2(z)/f_3(z)$$

lies in at least one of the circular regions C_1, C_2, C_3 or in a fourth circular region C. This fourth region is the locus of a point Z whose cross-ratio (22,3) with the points ζ_1, ζ_2 and ζ_3 has the constant value $(-m_2/m_1)$ as ζ_1, ζ_2 and ζ_3 describe the regions C_1, C_2 and C_3 respectively.

Regarding Th. (22,2), we may say that the same conclusion holds for the zeros of the derivative of the reciprocal $f_3(z)/f_1(z)f_2(z)$ of the above function. Furthermore, since the total "degree" of $f(z)$ is $n = n_1 + n_2 - n_3 = 0$, Th.

(22,2) may be restated in terms of the jacobian of the two binary forms (20,1), as is done in Walsh [1b, pp. 112-113].

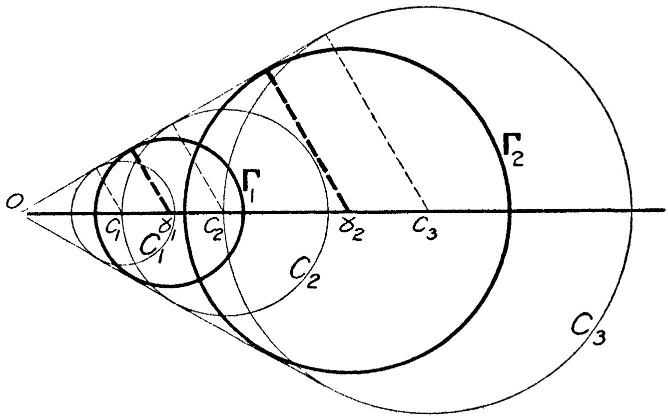

FIG. (22, 1)

Another case in which the p-circular $2p$-ic curve $E(z) = 0$ degenerates into a number of circles is the case that $m_i > 0$ and that the regions C_i are the interiors of circles having a common external center of similitude. (See fig. 22,1). The result in this case is due to Walsh [1c] and is embodied in

THEOREM (22,3). *If each zero of the polynomial $f_j(z)$ of degree n_j lies in the closed interior of a circle C_j and if the circles C_j have an external center of similitude O, then each zero of the derivative of the product $f(z) = f_1(z)f_2(z) \cdots f_p(z)$ lies either in the closed interior of one of the circles C_j ($j = 1, 2, \cdots, p$) or in the closed interior of one of the circles Γ_k ($k = 1, 2, \cdots, p-1$). The circles Γ_k have also the external center of similitude O; their centers are at the zeros of the logarithmic derivative of the polynomial*

$$(22,4) \qquad g(z) = (z - c_1)^{n_1}(z - c_2)^{n_2} \cdots (z - c_p)^{n_p}.$$

Let us verify this theorem in the case $p = 3$. Without loss of generality, we may take O at the origin and take the centers c_1, c_2 and c_3 of the circles on the x-axis. (See fig. 22,1.) The equation of each circle C_j has then the form

$$(22,5) \qquad C_j(z) = |z - c_j|^2 - (\lambda c_j)^2 = x^2 + y^2 - 2c_jx + \mu c_j^2, \qquad \mu = 1 - \lambda^2,$$

and the square of the common tangent of two such circles is

$$(22,6) \qquad \tau_{jk} = |c_j - c_k|^2 - \lambda^2(c_j - c_k)^2 = \mu(c_j - c_k)^2.$$

If eqs. (22,5) and (22,6) are substituted into eq. (21,4) after all subscripts have been increased by one, we obtain the equation

$$(22,7) \qquad n^2(x^2 + y^2) - 2Ax(x^2 + y^2) + B(x^2 + y^2) + 4Cx^2 - 2Dx + E = 0$$

where $m_i \equiv n_i$ and

$$A = n[(n - n_1)c_1 + (n - n_2)c_2 + (n - n_3)c_3],$$

$$B = \mu\{(n - n_1)^2c_1^2 + (n - n_2)^2c_2^2 + (n - n_3)^2c_3^2 + 2n_1n_2c_1c_2 + 2n_2n_3c_2c_3 + 2n_1n_3c_1c_3\},$$

$$C = n(n_3c_1c_2 + n_2c_1c_3 + n_1c_2c_3),$$

$$D = \mu\{n_3c_1c_2[(n - n_1)c_1 + (n - n_2)c_2] + n_2c_1c_3[(n - n_1)c_1 + (n - n_3)c_3] + m_1c_2c_3[(n - n_2)c_2 + (n - n_3)c_3] + 2(n_1n_2 + n_1n_3 + n_2n_3)c_1c_2c_3\},$$

$$E = \mu^2\{n_3^2c_1^2c_2^2 + n_2^2c_1^2c_3^2 + n_1^2c_2^2c_3^2 + 2c_1c_2c_3(n_2n_3c_1 + n_1n_3c_2 + n_1n_2c_3)\}.$$

On the other hand, the zeros of the logarithmic derivative of (22,4) in this case satisfy the equation

$$n_3(z - c_1)(z - c_2) + n_2(z - c_1)(z - c_3) + n_1(z - c_2)(z - c_3) = 0. \tag{22,8}$$

Denoting the roots of (22,8) by γ_1 and γ_2, we have the relations from eq. (22,8)

$$\begin{aligned} \gamma_1 + \gamma_2 &= \frac{1}{n}[(n - n_1)c_1 + (n - n_2)c_2 + (n - n_3)c_3], \\ \gamma_1\gamma_2 &= \frac{1}{n}[n_3c_1c_2 + n_2c_1c_3 + n_1c_2c_3]. \end{aligned} \tag{22,9}$$

The circles Γ_1 and Γ_2 with centers γ_1 and γ_2 and with O as center of similitude have the equations of form (22,5)

$$\Gamma_j(z) \equiv x^2 + y^2 - 2\gamma_jx + \mu\gamma_j^2 = 0. \tag{22,10}$$

Multiplying together these two equations, we obtain

$$\begin{aligned} \Gamma_1(z)\Gamma_2(z) \equiv (x^2 + y^2)^2 - 2(\gamma_1 + \gamma_2)x(x^2 + y^2) + \mu(\gamma_1^2 + \gamma_2^2)(x^2 + y^2) \\ + 4\gamma_1\gamma_2x^2 - 2\gamma_1\gamma_2\mu(\gamma_1 + \gamma_2)x + \mu^2\gamma_1^2\gamma_2^2 = 0. \end{aligned} \tag{22,11}$$

Using eqs. (22,9) and other symmetric functions of γ_1 and γ_2, we may show that eq. (22,11) is the same as that obtained by dividing eq. (22,7) by n^2. In other words, the bicircular quartic (22,7) degenerates into the two circles Γ_1 and Γ_2 as required in Th. (22,3) with $p = 3$.

Th. (22,3) may be generalized to rational functions of the form (21,2). If the regions C_j are the interiors of circles and if, exterior to all the circles C_j, there is a point P which is an external center of similitude for every pair C_i, C_j when i and j are both less than $k + 1$ or both greater than k, but which is an internal center of similitude for all other pairs C_i, C_j, then the curve

$E(z) = 0$ again degenerates into a set of circles with P as an internal or external center of similitude. For further details, the reader is referred to Walsh [1c, p. 45].

EXERCISES. Prove the following.

1. Th. (19,1) is a special case of Ths. (21,1) with $p = 1$ and (22,3) with $p = 2$ and of Th. (22,2) with region C_3 taken as the point at infinity.

2. Let the circles C_j ($j = 1, 2, \cdots, p$) have the collinear centers c_j and equal radii r. Let the polynomial $f_j(z)$ of degree n_j have all its zeros in the closed interior of C_j. Let C_j' denote the circles of radius r, which circles have their centers at the zeros of the logarithmic derivative of the $g(z)$ of eq. (22,4). Then the zeros of the derivative of the product $f(z) = f_1(z)f_2(z) \cdots f_p(z)$ lie in the point set consisting of the closed interiors of the circles C_j ($j = 1, 2, \cdots, p$) for which $n_j > 1$ and the circles C_j' ($j = 1, 2, \cdots, p - 1$). Hint: Use Th. (21,1) or allow point O in Th. (22,3) to recede to infinity [Walsh 1c, p. 53].

3. Let the r in ex. (22,2) be a sufficiently small number. Then the zeros of $f^{(k)}(z)$ lie in the point set consisting of the closed interiors of the circles C_j ($j = 1, 2, \cdots, p$) and C_j'' ($j = 1, 2, \cdots, p - k$), the latter being of radius r and having their centers at the zeros of $g^{(k)}(z)/g(z)$ [Walsh 1c, p. 53].

4. For $m_1 = m_2 = m_0$, $r_1 = r_2 = r_0$ and the centers c_j at the vertices of an equilateral triangle whose center is 0, bicircular quartic (21,4) degenerates into two circles concentric at 0, the larger of which has the radius $(r_1^2 + hr_1)^{1/2}$, h being the distance from 0 to c_j [Walsh 5].

5. For $m_j = m$, $r_j = r$ ($j = 1, 2, \cdots, p$) and the c_j as the roots of the equation $z^p = h^p$ where $0 < h$, the curve $E(z) = 0$ of Th. (21,1) (with all subscripts increased by one) degenerates into a set of circles concentric at $z = 0$ having as radii the roots R of the equation

$$\sum_1^p (3t_j) - \sum_{j=1,k=j+1}^{p} 4h^2 t_j t_k \sin^2 (\pi(j - k)p^{-1}) = 0,$$

where $1/t_j = R^2 + h^2 - r^2 - 2hR \cos (2\pi jp^{-1})$ [Marden 3, p. 98].

6. Let $a, b, c_1, c_2, c_3, \cdots$ be real numbers and let each z_j be a point in or on the circle C_j with center at c_j and with radius r. Then the zeros of the derivative of the entire function of genre zero or one

$$f(z; z_1, z_2, z_3, \cdots) = \exp (az + b) \prod_{k=1}^{\infty} (1 - z/z_k)$$

lie in the circles C_j' of radius r with centers at the zeros of the derivative of $f(z; c_1, c_2, c_3, \cdots)$ [Walsh 11].

7. Let $C_1, C_2, \cdots, C_p$ be circles of equal radius r with centers at the collinear points $c_1, c_2, \cdots, c_p$. Assume in eq. (9,1) that the α_j are positive and denote by $s(z)$ a Stieltjes polynomial corresponding to $a_j = c_j$, $j = 1, 2, \cdots, p$. Let $C_1', C_2', \cdots, C_n'$ denote the circles of radius r with centers at the zeros of $s(z)$. If no circle C_j' has a point in common with any other C_j' or with any circle C_j,

then the locus of the zeros of the Stieltjes polynomial $S(z)$ as the point a_j varies over the closed interior of the circle C_j $(j = 1, 2, \cdots, p)$ is the region consisting of the closed interiors of the circles C_j' $(j = 1, 2, \cdots, n)$. Furthermore, each C_j' contains just one zero of $S(z)$ [Walsh 8].

8. The curve (21,16) reduces to one or more circles in the following cases: (a) $p = 1$ (cf. Cor. 18,1); (b) λ real and the regions C_j taken as the closed interiors of equal circles with centers on a line parallel to the axis of reals [Walsh 9].

CHAPTER VI

THE CRITICAL POINTS OF A POLYNOMIAL WHICH HAS ONLY SOME PRESCRIBED ZEROS

23. Polynomials with two given zeros. In Chapters II and V we developed several theorems on the location of *all* the critical points of a polynomial $f(z)$ when the location of *all* the zeros of $f(z)$ is known. In the present chapter we shall investigate the extent to which the prescription of only *some* of the zeros of $f(z)$ fixes the location of *some* of the critical points of $f(z)$.

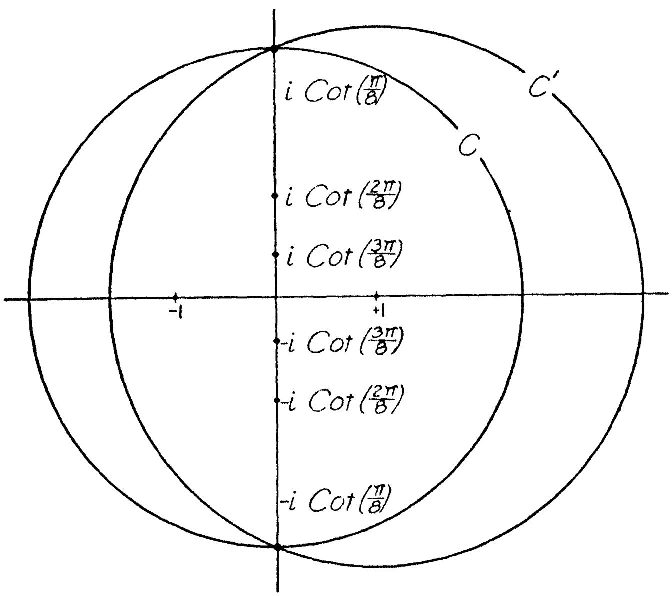

FIG. (23, 1)

A first result of this nature is the one which we may derive immediately from Rolle's Theorem by using eq. (10,7) and thereby transforming the real axis into an arbitrary line L. This result states that, if the zeros of a polynomial are symmetric in a line L, then between any pair of zeros lying on L may be found at least one zero of the derivative.

We now ask whether or not, given two zeros of a polynomial, we may determine the location of a zero of $f'(z)$ even when no additional hypothesis (such as that of symmetry in a line) is placed upon the remaining zeros. An affirmative

answer to this question, as given first by Grace [1] and later by Heawood [1], is stated in the

GRACE-HEAWOOD THEOREM (Th. 23,1). *If z_1 and z_2 are any two zeros of an nth degree polynomial $f(z)$, at least one zero of its derivative $f'(z)$ will lie in the circle C with center at* $[(z_1 + z_2)/2]$ *and with a radius of* $[(1/2) \mid z_1 - z_2 \mid (\cot \pi/n)]$.

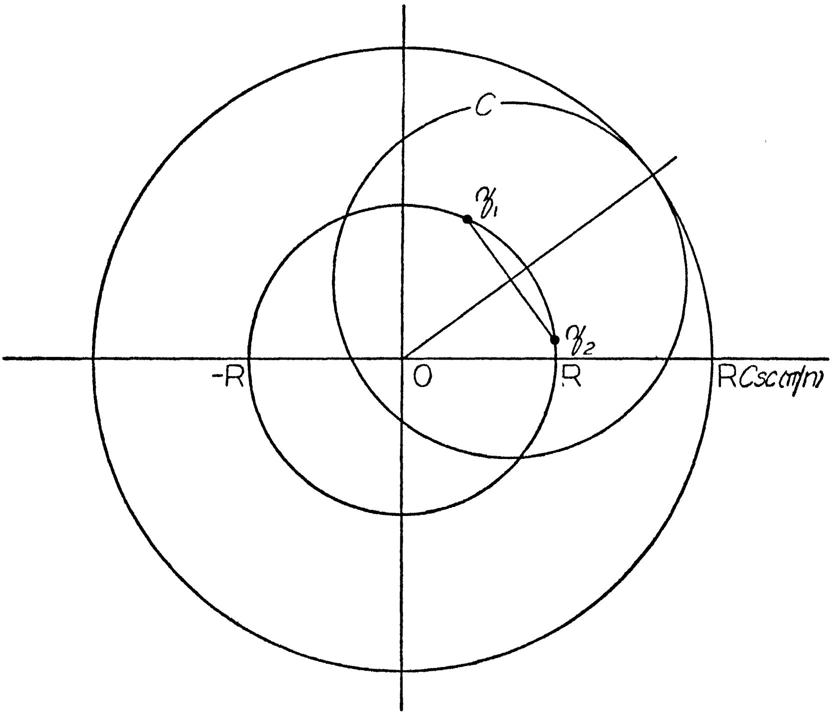

FIG. (23, 2)

In proving this theorem we may without loss of generality take $z_1 = +1$ and $z_2 = -1$. (See fig. (23,1) for case $n = 8$.) By hypothesis, we have upon

$$f'(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-2} z^{n-2} + z^{n-1}$$

the requirement that

$$(23,1) \qquad 0 = f(1) - f(-1) = \int_{-1}^{+1} f'(t)\, dt = 2a_0 + \frac{2a_2}{3} + \frac{2a_4}{5} + \cdots .$$

Since eq. (23,1) is a linear relation among the coefficients of $f'(z)$, we may apply Th. (15,3). Thus at least one zero of $f'(z)$ lies in every circular region containing the zeros of the polynomial

$$g(z) = \int_{-1}^{+1} (z - t)^{n-1}\, dt = (1/n)[(z - 1)^n - (z + 1)^n].$$

But the zeros of $g(z)$ are $z_k = -i \cot (k\pi/n)$, $k = 1, 2, \cdots, n-1$. This means not only that at least one zero of $f'(z)$ lies in the circle C of Th. (23,1) but also that at least one zero of $f'(z)$ lies in every circle C' (see fig. 23,1) through the two points $z = \pm i \cot \pi/n$.

That the radius r of the Grace-Heawood Theorem may not be replaced by a smaller number may be seen from the polynomial

$$f(z) = \int_{-1}^{z} (t - i \cot \pi/n)^{n-1}\, dt$$

$$= \frac{1}{n} \{[z - i \cot (\pi/n)]^n - [-1 - i \cot (\pi/n)]^n\}$$

which has the zeros $z = \pm 1$ and the derivative of which,

$$f'(z) = [z - i \cot (\pi/n)]^{n-1},$$

has its only zero at $z = i \cot (\pi/n)$.

Let us now allow the points z_1 and z_2 to vary arbitrarily within circle $|z| \leqq R$ and inquire regarding the envelope (see fig. 23,2) of the corresponding circle C of Th. (23,1). It is clearly sufficient to consider the envelope of the circle when $|z_1| = |z_2| = R$. Any point on the circumference of C may be represented by the complex number:

$$(23,2) \qquad \zeta = \frac{z_1 + z_2}{2} + e^{i\omega} \frac{|z_1 - z_2|}{2} \cot \left(\frac{\pi}{n}\right).$$

Corresponding to point ζ, two points z_1 and z_2 on the circle $|z| = R$ may be found so that either $z_1 = z_2 e^{i\psi}$ or $z_2 = z_1 e^{i\psi}$ with $0 \leqq \psi \leqq \pi$ and so that eq. (23,2) is satisfied. Thus,

$$|\zeta| \leqq (R/2)\,|1 + e^{i\psi}| + (R/2)\,|1 - e^{i\psi}| \cot (\pi/n),$$

$$|\zeta| \leqq R[\cos (\psi/2) + \sin (\psi/2) \cot (\pi/n)]$$

$$\leqq R \sin (\pi/n + \psi/2) \csc (\pi/n) \leqq R \csc (\pi/n).$$

We have thereby proved the following result due to Alexander [1], Kakeya [2] and Szegö [1].

THEOREM (23,2). *If two zeros of an nth degree polynomial lie in or on a circle of radius R, at least one zero of its derivative lies in or on the concentric circle of radius $R \csc (\pi/n)$.*

This is again the best result, as may be seen by choosing

$$f(z) = \int_{+1}^{z} [-2i \csc (2\pi/n) + u \exp (-\pi i/n) \cos (\pi/n)]^{n-1}\, du.$$

For, this polynomial has on the unit circle the zeros $z_1 = +1$ and $z_2 = e^{2\pi i/n}$

and furthermore its derivative has a zero on the circle $|z| = \csc \pi/n$ at the point $z = e^{\pi i/n} \csc (\pi/n)$.

EXERCISES. Prove the following.

1. If $f'(z) \neq 0$ in $|z| \leqq r$, $f(z)$ has at most one zero in $|z| \leqq r \sin (\pi/n)$. Hint: Use Th. (23,2).

2. If the derivative $f'(z)$ of an nth degree polynomial $f(z)$ is different from zero in a circle C' of radius r, then $f(z)$ cannot assume any value A twice in the concentric circle C'' of radius $r \sin \pi/n$. In other words, $f(z)$ is *univalent* in C'. Hint: Apply ex. 1 to $f(z) - A$ [Alexander 1, Kakeya 2, Szegö 1].

24. **Mean-value theorems.** We may derive results similar to those of sec. 23 on using the following Mean-Value Theorems. In the form stated below, these theorems were first proved by Marden [7] and [8], but in certain special cases they had been previously treated in Fekete [2 − 6] and Nagy [4]. Both theorems employ the notation $S(K, \phi)$ as in sec. 8 for the star-shaped region comprised of all points from which the convex region K subtends an angle of at least ϕ.

THEOREM (24,1). *Let $P(z)$ be an nth degree polynomial and let z_1, z_2, $\cdots$, z_m be any m points of a convex region K. Let σ, the mean-value of $P(z)$ in the points z_j, be defined by the equation*

$$\sigma \sum_{j=1}^{m} \alpha_j = \sum_{j=1}^{m} \alpha_j P(z_j) \tag{24,1}$$

where

$$\mu \leqq \arg \alpha_j \leqq \mu + \gamma < \mu + \pi, \qquad j = 1, 2, \cdots, m.$$

Then the star-shaped region $S(K, (\pi - \gamma)/n)$ contains at least one point s at which $P(s) = \sigma$.

If H denotes the smallest convex region of the w-plane containing the points $w = P(z_1), P(z_2), \cdots, P(z_m)$, then, according to Th. (8,1) applied to $F(w) = \sum \alpha_j(w - P(z_j))$, σ is an arbitrary point of the region $S(H, \pi - \gamma)$.

THEOREM (24,2). *Let $P(z)$ be an nth degree polynomial and let $C : z = \psi(t)$ (t real; $a \leqq t \leqq b$) be a rectifiable curve drawn in a convex region K. On curve C, let $\alpha(t)$ be a continuous function whose argument satisfies the inequality*

$$\mu \leqq \arg \alpha(t) \leqq \mu + \gamma < \mu + \pi.$$

Then, the star-shaped region $S(K, (\pi - \gamma)/n)$ contains at least one point s for which

$$\int_a^b P[\psi(t)]\alpha[\psi(t)]\, dt = P(s) \int_a^b \alpha[\psi(t)]\, dt. \tag{24,2}$$

Ths. (24,1) and (24,2) could be combined into a single theorem if Stieltjes integrals were introduced into eq. (24,2).

The proofs of both Theorems (24,1) and (24,2) are essentially the same.

For example, to prove the first, let us write eq. (24,1) in the form

$$\sum_{j=1}^{m} \alpha_j[P(z_j) - \sigma] = 0. \tag{24,3}$$

Denoting by $a_1, a_2, \cdots, a_n$ the points at which $P(z)$ assumes the value σ, we may set up the equation

$$P(z) - \sigma = A(z - a_1)(z - a_2) \cdots (z - a_n). \tag{24,4}$$

If every a_k were to lie exterior to $S(K, (\pi - \gamma)/n)$, the region K would subtend in each a_k an angle less than $(\pi - \gamma)/n$. That is, a constant δ_k could be found so that

$$0 \leqq \arg (z_j - a_k) - \delta_k < (\pi - \gamma)/n, \quad k = 1, 2, \cdots, n; j = 1, 2, \cdots, m. \tag{24,5}$$

Adding these inequalities for $k = 1, 2, \cdots, n$ and substituting from eq. (24,4), we conclude that

$$0 \leqq \arg [P(z_j) - \sigma] - \arg A - \sum_{k=1}^{n} \delta_k < \pi - \gamma, \qquad j = 1, 2, \cdots, m.$$

Hence, by Th. (1,1)

$$\sum_{j=1}^{m} \alpha_j[P(z_j) - \sigma] \neq 0$$

in contradiction to eq. (24,3).

We shall now apply Th. (24,2) to the determination of the zeros of $P(z)$; that is, the points s for which $P(s) = 0$. Since $\int_a^b \alpha[\psi(t)]\, dt \neq 0$ in Th. (24,2), we deduce at once a result which for $\gamma = 0$ is due to Fekete [5] and [6] and for γ arbitrary is due to Marden [7].

THEOREM (24,3). *Let $P(z)$ be an nth degree polynomial; let C: $z = \psi(t)$ (t real; $a \leqq t \leqq b$) be a rectifiable curve drawn in a convex region K and let $\alpha(t)$ be on C a continuous function whose argument satisfies the inequality*

$$\mu \leqq \arg \alpha(z) \leqq \mu + \gamma < \mu + \pi.$$

Then, if

$$\int_a^b P[\psi(t)]\alpha[\psi(t)]\, dt = 0, \tag{24,6}$$

$P(z)$ has at least one zero in the star-shaped region $S(K, (\pi - \gamma)/n)$.

As an application of Th. (24,3), let us choose

$$\gamma = 0, \qquad \alpha(z) \equiv 1, \qquad a = 0, \qquad b = 1,$$

$$\psi(t) = (1 - t)\xi + t\eta.$$

Let us denote by $Q(z)$ an nth degree polynomial which assumes the same values

at the points ξ and η. If now we replace n by $n - 1$ in Th. (24,3) and if we set $P(z) \equiv Q'(z)$, then we find

$$\int_0^1 Q'[(1 - t)\xi + t\eta]\, dt = [Q(\xi) - Q(\eta)]/(\xi - \eta) = 0.$$

That is, eq. (24,6) is satisfied. Hence, at least one zero of $Q'(z)$ lies in $S(K, \pi/(n - 1))$. Since K may be taken as the line-segment joining the points ξ and η, we have established the following result due to Fekete [5].

THEOREM (24,4). *If the nth degree polynomial $P(z)$ has the two zeros $z = \xi$ and $z = \eta$, its derivative will have at least one zero in the region comprised of all points from which the line-segment $\xi\eta$ subtends an angle of at least $[\pi/(n - 1)]$.*

EXERCISES. Prove the following.

1. Let $a \neq b$, $A \neq B$, $0 < \phi \leqq \pi$, and $C \neq A$, $C \neq B$, $|\arg (C - B)/(C - A)| \geqq \phi$. If an nth degree polynomial $P(z)$ assumes the value A at $z = a$ and B at $z = b$, it assumes the value C at least once in S (segment ab, ϕ/n) [Fekete 7]. Remark: For $\phi = \pi$, this result is analogous to the Bolzano Theorem that, if a real continuous function $f(x)$ of the real variable x assumes the value A at $x = a$ and the value $B \neq A$ at $x = b \neq a$, it assumes every value between A and B at least once on the line-segment $a < x < b$.

2. Let $C : z = \psi(t)$ (t real, $a \leqq t \leqq b$) be a rectifiable curve drawn in a convex region K and let $\alpha(z)$ be a function which is continuous on C and assumes on C only values in a sector A with vertex at the origin and with an opening $\gamma < \pi$. Let p and q be positive integers with $m = \max (p, q)$ and let S be the star-shaped region consisting of all points from which K subtends an angle of at least $(\pi - \gamma)/(m + q)$. Finally, let $P(z)$ and $Q(z)$ be polynomials of degree p and q respectively such that $R(z) = P(z)/Q(z)$ is irreducible and has no poles in S. Then in S there exists at least one point s such that for $z = \psi(t)$

$$\int_a^b R(z)\alpha(z)\, dt = R(s) \int_a^b \alpha(z)\, dt \qquad \text{[Marden 7].}$$

25. **Polynomials with p known zeros.** As a generalization of Ths. (23,2) and (24,4), we shall now consider the problem: given that p zeros of a polynomial $f(z)$ of degree n ($n \geqq p$) lie in a circle C of radius R, to find the radius R' of the smallest concentric circle C' which contains at least $p - 1$ zeros of the derivative $f'(z)$.

This problem was first proposed by Kakeya [1]. He showed that there exists a function $\phi(n, p)$ such that $R' = R\phi(n, p)$. Furthermore, as in Th. (23,2), he established the result that $\phi(n, 2) = \csc (\pi/n)$, but did not succeed in obtaining an explicit formula or an estimate for $\phi(n, p)$ for other values of p.

Subsequently, Biernacki [1] derived an estimate for $\phi(n, p)$ in the case $p = n - 1$; namely, $\phi(n, n - 1) \leqq (1 + 1/n)^{1/2}$.

In order to throw light upon the general question, we shall, as in Marden [11], first extend Th. (24,4) to polynomials having a given pair of multiple zeros.

THEOREM (25,1). *If z_1 and z_2 are respectively k_1-fold and k_2-fold zeros of an nth degree polynomial $f(z)$, then at least one zero (different from z_1 and z_2) of the derivative lies in the circle C with center at the point $[(z_1 + z_2)/2]$ and with a radius $[(1/2) \mid z_1 - z_2 \mid \cot (\pi/2q)]$, where $q = n + 1 - k_1 - k_2$.*

In proving this theorem we suffer no loss of generality in taking $z_1 = -1$ and $z_2 = +1$. Let us apply Theorem (24,3), choosing

$$\alpha(z) = (1 + z)^{k_1-1}(1 - z)^{k_2-1}, \qquad P(z) = f'(z)/\alpha(z), \qquad \psi(t) \equiv t.$$

For these choices $\arg \alpha(z) = 0 = \gamma$ on the straight line $z = \psi(t)$, $-1 \leqq t \leqq 1$, and $P(z)$ is a polynomial of degree q. According to Th. (24,3) at least one zero of $P(z)$ lies in the star-shaped region comprised of all points at which the segment $-1 \leqq z \leqq 1$ subtends an angle not less than π/q. The smallest circle which encloses the latter region is clearly the circle described in Th. (25,1).

We now ask: what is the envelope of the circle C of Th. (25,1) when the points z_1 and z_2 vary independently over a circle of radius R? To answer this question, we may employ the method used in the proof of Th. (23,2). We thus obtain the following result due to Marden [11].

THEOREM (25,2). *If a circle C of radius R contains a k_1-fold zero and a k_2-fold zero of an nth degree polynomial, then the concentric circle C' of radius $R \csc \pi/2q$, $q = n + 1 - k_1 - k_2$, contains zeros of the derivative with a total multiplicity of at least $k_1 + k_2 - 1$.*

In order to generalize this theorem to the case that the circle C contains p zeros which are not necessarily concentrated at just two points, we shall employ the following identity which connects any p zeros of a polynomial with any $q = n - p + 1$ zeros of its derivative.

THEOREM (25,3). *Among the $n + 1$ distinct numbers $\alpha_1, \alpha_2, \cdots, \alpha_p, \beta_1, \beta_2, \cdots, \beta_q$, $p + q = n + 1$, let the α_j be zeros of an nth degree polynomial and let the β_k be zeros of its derivative. Then*

$$\sum \frac{D_{j_1 j_2 \cdots j_q}}{(\beta_1 - \alpha_{j_1})(\beta_2 - \alpha_{j_2}) \cdots (\beta_q - \alpha_{j_q})} = 0 \tag{25,1}$$

where $j_1, j_2, \cdots, j_q$ run independently through the values $1, 2, \cdots, p$ and where

$$D_{j_1 j_2 \cdots j_q} = \prod_{m=1}^{p} (\delta_{m j_1} + \delta_{m j_2} + \cdots + \delta_{m j_q})!$$

with $\delta_{mj} = 1$ or 0 according as $j = m$ or $j \neq m$.

This identity, which is due to Marden [11], is a generalization of the formula

$$\sum_{j=1}^{m} \frac{1}{\beta_k - \alpha_j} = 0 \tag{25,2}$$

which connects any one zero β_k of $f'(z)/f(z)$ with the n zeros $\alpha_1, \alpha_2, \cdots, \alpha_n$ of $f(z)$.

To establish Theorem (25,3), it is necessary to eliminate the $q - 1$ numbers $\alpha_{p+1}, \alpha_{p+2}, \cdots, \alpha_n$ from the q equations (25,2), $k = 1, 2, \cdots, q$. As this elimination is quite involved, we shall omit the details and proceed immediately to the proof of a result due to Marden [11].

THEOREM (25,4). *If a circle C of radius R contains p zeros of an nth degree polynomial $f(z)$, the concentric circle C' of radius R csc $(\pi/2q)$, $q = n - p + 1$, contains at least $p - 1$ zeros of the derivative $f'(z)$.*

Let us suppose, on the contrary, that at most $p - 2$ zeros of $f'(z)$ lie in or on C' and that, hence, $(n - 1) - (p - 2) = q$ zeros of $f'(z)$ lie outside C'. Let us denote these zeros of $f'(z)$ by $\beta_1, \beta_2, \cdots, \beta_q$ and let us denote the zeros of $f(z)$ lying in C by $\alpha_1, \alpha_2, \cdots, \alpha_p$. Obviously, no α_j, $1 \leqq j \leqq p$, may equal a β_k, $1 \leqq k \leqq q$. At each β_k, the circle C subtends an angle less than π/q. This means that a point ξ_k on C may be found so that

$$0 \leqq \arg \frac{\xi_k - \beta_k}{\alpha_j - \beta_k} < \pi/q, \qquad j = 1, 2, \cdots, p. \tag{25,3}$$

If now the sum in eq. (25,1) be multiplied by $(\beta_1 - \xi_1)(\beta_2 - \xi_2) \cdots (\beta_q - \xi_q)$, the resulting sum would have the terms of the form

$$\frac{(\xi_1 - \beta_1)(\xi_2 - \beta_2) \cdots (\xi_q - \beta_q)}{(\alpha_{j_1} - \beta_1)(\alpha_{j_2} - \beta_2) \cdots (\alpha_{j_q} - \beta_q)}. \tag{25,4}$$

Since each term (25,4) would, because of (25,3), be representable by a vector in the angle

$$0 \leqq \arg z < \pi,$$

we know by Th. (1,1) that the sum cannot vanish. This result, being in contradiction to eq. (25,1), shows that at least $p - 1$ zeros of $f'(z)$ must lie in circle C'.

EXERCISES. Prove the following.

1. If a convex region K contains p zeros of an nth degree polynomial $f(z)$, the star-shaped region $S(K, \pi/q)$, $q = n - p - 1$, contains at least $p - 1$ zeros of the derivative $f'(z)$. Hint: Use Th. (25,3) [Marden 11].
2. If $f'(z)$ has at most $p - 1$ zeros in a circle of radius ρ, $f(z)$ has at most p zeros in the concentric circle of radius $\rho \sin [\pi/2(n - p)]$. Hint: Assume the contrary [Marden 11].
3. If the derivative of an nth degree polynomial $f(z)$ has at most $p - 1$ zeros in a circle C of radius ρ, then $f(z)$ assumes no value A more than p times in the concentric circle C' of radius $\{\rho \sin [\pi/2(n - p)]\}$; that is, $f(z)$ is at most *p-valent* in C'. Hint: Apply ex. 2 to $f(z) - A$ [Marden 11].
4. The polynomial

$$f(z) = \left[z^2 - 2z\left(\frac{n}{2n - p}\right)^{1/2} + 1\right]^{p/2}\left[z - \frac{1}{p}(n(2n - p))^{1/2}\right]^{n-p}$$

with p a positive even integer has two zeros on the unit circle, each of multiplicity $(p/2)$. Its derivative has zeros at the same points each of multiplicity $(p-2)/2$ and has a double zero at the point $z = (2 - p/n)^{1/2}$. Thus, for p even the $\phi(n, p)$ defined at the beginning of sec. 25 satisfies the inequality

$$\phi(n, p) \geqq (2 - p/n)^{1/2} \tag{25,5}$$ [Marden 11].

5. Let $Z_1, Z_2, \cdots, Z_p$ be zeros of the function

$$F(z) = \sum_{j=0}^{p-1} A_j z^j + \sum_{j=0}^{n} m_j/(z - z_j),$$

where the A_j are arbitrary complex constants. Then

$$F(z) = \sum_{j=0}^{n} \left\{[m_j/(z - z_j)] \prod_{k=1}^{p} [(Z_k - z)/(Z_k - z_j)]\right\}. \tag{25,6}$$

Hint: Eliminate the A_j from $F(z)$ by using the eqs. $F(Z_k) = 0, k = 1, 2, \cdots, p$. [Marden 19].

6. If in ex. (25,5) all the poles z_j lie in a convex region K and if the m_j are points in a convex sector with vertex at the origin and with an aperture μ, then at most p zeros of the function $F(z)$ lie exterior to the star-shaped region $S(K, (\pi - \mu)/(p + 1))$. Hint: Assume the contrary and consider the argument of each term in the eq. $F(Z_{p+1}) = 0$ obtained from eq. (25,6) with Z_1, $Z_2, \cdots, Z_{p+1}$ all taken exterior to S [Marden 19].

7. With $n = \infty$ the results in exs. (25,5) and (25,6) are valid for the meromorphic function

$$M(z) = \sum_{j=0}^{p-1} B_j z^j + [m_0/(z - z_0)] + \sum_{j=1}^{\infty} m_j\left\{[1/(z - z_j)] + \sum_{k=1}^{p-1} (z/z_j)^k\right\},$$

where the B_j are arbitrary complex constants, if $\sum_{j=1}^{\infty} | m_j |/| z_j |^p$ converges. Hint: In ex. (25,5) take $A_j = B_j + \sum_{k=1}^{n} m_k(1/z_k)^j$. Then, as $n \to \infty$, $F(z) \to M(z)$ absolutely and uniformly in every finite closed region not containing any z_j [Marden 19].

8. Let $z_0 = 0, z_1, z_2, \cdots$ be the zeros of an entire function $E(z)$ of genre p so that $E(z)$ may be written in the Weierstrass form

$$E(z) = e^{P(z)} z^{m_0} \prod_{j=1}^{\infty} \left\{(1 - z/z_j) \exp \sum_{k=1}^{p} [(z/z_j)^k/k]\right\}.$$

Then, if $Z_1, Z_2, \cdots, Z_p$ are any p zeros of $E'(z)$, and if $m_j = 1$ for $1 \leqq j$,

$$E'(z) = E(z) \sum_{j=0}^{\infty} \left\{[m_j/(z - z_j)] \prod_{k=1}^{p} [(Z_k - z)/(Z_k - z_j)]\right\}.$$

If all the zeros of $E(z)$ lie in a convex infinite region K, at most p zeros of $E'(z)$ lie exterior to the region $S(K, \pi/(p + 1))$. Hint: Apply ex. (25,7) to $E'(z)/E(z)$ [Marden 18].

26. Alternative treatment. As in sec. 25 let us denote by R the radius of a circle containing p zeros of an nth degree polynomial $f(z)$ and by R' the radius of the concentric circle containing at least $p - 1$ zeros of $f'(z)$. We shall now obtain another upper bound on R', this time by using ex. (19,4) and ex. (19,5) and induction.

As a first step, we shall prove

THEOREM (26,1). *If an nth degree polynomial $f(z)$ has p zeros in or on a circle C of radius R and an $(n - p)$-fold zero at a point ζ, then its derivative has at least $p - 1$ zeros in the concentric circle C' of radius $R' = R[(3n - 2p)/n]$.*

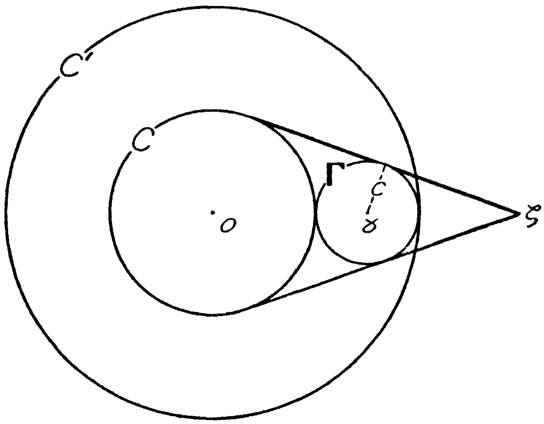

FIG. (26, 1)

Without loss of generality in the proof, we assume that C is the unit circle $|z| = 1$.

If $|\zeta| \leqq 1$, then all the zeros of $f(z)$ lie in circle C and by Th. (6,2) all $n - 1$ zeros of $f'(z)$ lie in C. In such a case, surely $p - 1$ zeros of $f'(z)$ lie in C'.

If $|\zeta| > 1$, ex. (19,5) informs us that (see fig. (26,1)) the zeros of $f'(z)$ lie in circle C and in a circle Γ with center $\gamma = p\zeta/n$ and radius $c = (n - p)/n$. If C and Γ do not overlap one another, exactly $p - 1$ zeros of $f'(z)$ lie in C and hence in C'. If C and Γ do overlap, but if Γ does not enclose ζ, precisely p zeros of $f'(z)$ lie in the region comprised of C and Γ and hence in the circle C' with center at the origin and radius

$$1 + [2(n - p)/n] = (3n - 2p)/n.$$

Finally, if C and Γ overlap and if Γ contains ζ, then all the zeros of $f(z)$ lie in C' and hence all $n - 1$ zeros of $f'(z)$ lie in C'.

In all cases, therefore, circle C' contains at least $p - 1$ zeros of $f'(z)$.

Let us now consider polynomials which have p zeros in a circle C, but do not have the remaining zeros necessarily concentrated at a single point. For such polynomials, we shall prove a result due to Biernacki [3].

THEOREM (26,2). *If an nth degree polynomial $f(z)$ has p ($p \leqq n$) zeros in a circle C of radius R, its derivative has at least $p - 1$ zeros in the concentric circle C' of radius*

$$R' = R \prod_{k=1}^{n-p} [(n + k)/(n - k)]. \tag{26,1}$$

Our proof will use the method of mathematical induction. Without loss of generality, we may assume that C has its center at the origin and that the zeros α_j of $f(z)$ have been labelled in the order of increasing modulus

$$|\alpha_1| \leqq |\alpha_2| \leqq \cdots \leqq |\alpha_n|. \tag{26,2}$$

We begin with the case $p = n - 1$. Since only one zero α_n is exterior to C, we learn from Th. (26,1) that at least $p - 1$ zeros of $f'(z)$ lie in the circle

$$|z| \leqq \left(1 + \frac{2}{n}\right)R < \left(1 + \frac{2}{n-1}\right)R = \left(\frac{n+1}{n-1}\right)R;$$

that is, in the circle C' with the radius R' as given by eq. (26,1) for $p = n - 1$.

Let us now suppose that Th. (26,2) has been verified for the cases $p = n - 1$, $n - 2, \cdots, N + 1$ and let us proceed to the case $p = N$.

If in this case $|\alpha_{N+1}| > (2n - N)R/N$, we may apply ex. (19,4) with $n_1 = N$, $n_2 = n - N$, $r_1 = R$ and $r_2 > (2n - N)R/N$. We thus find that

$$r > (1/n)\{N[(2n - N)R/N] - (n - N)R\} = R$$

and that $f'(z)$ has exactly $N - 1$ zeros in the circle $C_1 \equiv C$ and hence in the circle C'.

If, on the other hand, $|\alpha_{N+1}| \leqq (2n - N)R/N$, the circle $C : |z| \leqq \rho = (2n - N)R/N$ contains the $N + 1$ zeros $\alpha_1, \alpha_2, \cdots, \alpha_{N+1}$. Hence, according to Th. (26,2) applied with ρ replacing R and $N + 1$ replacing p, at least N zeros of $f'(z)$ lie in the circle

$$|z| \leqq \left[\frac{(2n - N)}{N}\right]\left[\frac{(n+1)}{(n-1)} \frac{(n+2) \cdots (2n - N - 1)}{(n-2) \cdots (N+1)}\right]R. \tag{26,3}$$

But this is the circle $C' : |z| \leqq R'$, with the R' given by eq. (26,1) for $p = N$.

In all cases in which $p = N$, there are therefore at least $N - 1$ zeros of $f'(z)$ in the circle C'. In other words, Th. (26,2) has been established by mathematical induction.

However, neither Th. (25,4) nor Th. (26,2) gives the complete answer to the question raised at the beginning of sec. 25. For, as a critical examination of their proofs will reveal, neither theorem gives in general the least number $\phi(n, p)$ with the property: if p zeros of $f(z)$ lie in a circle of radius R, then at least $p - 1$ zeros of $f'(z)$ lie in the concentric circle of radius $R\phi(n, p)$.

EXERCISES. Prove the following.

1. If the derivative of an nth degree polynomial $f(z)$ has at most $p - 1$ zeros

in a circle C of radius ρ, then $f(z)$ has at most p zeros in the concentric circle C' of radius

$$\rho' = \rho \prod_{k=1}^{n-p-1} [(n-k)/(n+k)]. \tag{26,4}$$

2. The $f(z)$ of ex. (26,1) is at most p-valent in $|z| \leq \rho'$.

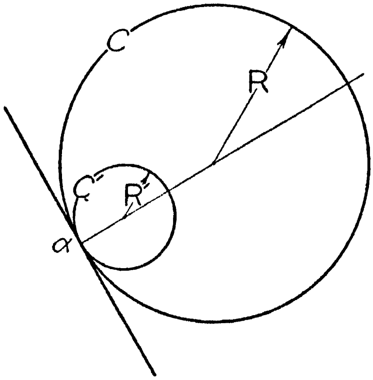

FIG. (26,2)

3. Th. (6,2) is the special case $p = n$ of both Ths. (25,4) and (26,2).
4. In the case $p = 2$, Th. (23,2) is better than Ths. (25,4) and (26,2).
5. Let α be a p-fold zero of the nth degree polynomial $f(z)$ and let C, a circle of radius R, pass through α but not contain any other zeros of $f(z)$. Let C', a circle of radius $R' = (p/n)R$, be tangent to C internally at α. (See fig. 26,2.) Then $f'(z) \neq 0$ in C'. Hint: Apply ex. (19,5).

Remark. The radius R' may, as shown in Nagy [5], be replaced by the larger number $R'' = p/(S + p)$, where S is the maximum number of zeros of $f(z)$ to either side of the line tangent to C at α.

CHAPTER VII

BOUNDS FOR THE ZEROS AS FUNCTIONS OF ALL THE COEFFICIENTS

27. The moduli of the zeros. So far we have studied the location of the zeros of the derivative of a polynomial $f(z)$ relative to the zeros of $f(z)$. The results which we obtained led to corresponding results concerning the relative location of the zeros of various other pairs of polynomials. In short, we may regard the preceding six chapters as concerned with the investigation of the zeros $Z_1, Z_2, \cdots, Z_m$ of a polynomial $F(z)$ as functions $Z_k = Z_k(z_1, z_2, \cdots, z_n)$ of some or all of the zeros z_k of a related polynomial $f(z)$.

In the remaining four chapters, our interest will be centered upon the study of the zeros z_k of a polynomial

$$(27,1) \qquad f(z) = a_0 + a_1 z + \cdots + a_n z^n$$

as functions $z_k = z_k(a_0, a_1, \cdots, a_n)$ of some or of all the coefficients a_j of $f(z)$. Our problems will fall mainly into two categories:

(I) Given an integer p, $1 \leqq p \leqq n$; to find a region $R = R(a_0, a_1, \cdots, a_n)$ containing at least or exactly p zeros of $f(z)$. For instance, we shall try to find the smallest circle $|z| = r$ which will enclose the p zeros.

(II) Given a region R, to find the number $p = p(a_0, a_1, \cdots, a_n)$ of zeros in R. An example of such a problem is that of finding the number p of zeros whose moduli do not exceed some prescribed value r.

While the regions R to be considered will be largely the circular regions, usually half-planes and the interiors of circles, we shall also consider other regions R such as sectors and annular rings.

Just as some of the preceding results were complex-variable analogues of Rolle's Theorem, so will some of the succeeding results, particularly those connected with the problems of the second category, be complex-variable analogues of the rules of sign of Descartes and Sturm.

Let us begin with a problem of the first category: to find an upper bound for the moduli of all the zeros of a polynomial. A classic solution of such a problem is the result due to Cauchy [1]; namely,

THEOREM (27,1). *All the zeros of the polynomial $f(z) = a_0 + a_1 z + \cdots + a_n z^n$, $a_n \neq 0$, lie in the circle $|z| \leqq r$, where r is the positive root of the equation*

$$(27,2) \qquad |a_0| + |a_1| z + \cdots + |a_{n-1}| z^{n-1} - |a_n| z^n = 0.$$

Obviously, the limit is attained when $f(z)$ is the left-side of (27,2).

The proof hinges on the inequality, obtained from eq. (27,1),

$$(27,3) \qquad |f(z)| \geqq |a_n||z|^n - (|a_0| + |a_1||z| + \cdots + |a_{n-1}||z|^{n-1}).$$

If $|z| > r$, the right side of (27,3) is positive since the left side of eq. (27,2) is negative for $r < z \leqq +\infty$. Hence $f(z) \neq 0$ for $|z| > r$.

From inequality (27,3), there follows immediately a second result also due to Cauchy [1]; namely,

THEOREM (27,2). *All the zeros of* $f(z) = a_0 + a_1 z + \cdots + a_n z^n$, $a_n \neq 0$, *lie in the circle*

$$(27,4) \qquad |z| < 1 + \max |a_k/a_n|, \qquad k = 0, 1, 2, \cdots, n-1.$$

For, if $M = \max |a_k/a_n|$ and if $|z| > 1$, we may infer from ineq. (27,3) that

$$\begin{aligned} |f(z)| &\geqq |a_n||z|^n\left\{1 - M\sum_{j=1}^{n}|z|^{-j}\right\} \\ &> |a_n||z|^n\left\{1 - M\sum_{j=1}^{\infty}|z|^{-j}\right\} \\ &> |a_n||z|^n\left\{1 - \frac{M}{|z|-1}\right\} = |a_n||z|^n\left\{\frac{|z|-1-M}{|z|-1}\right\}. \end{aligned}$$

Hence, if $|z| \geqq 1 + M$, then $|f(z)| > 0$. That is, the only zeros of $f(z)$ in $|z| > 1$ are those satisfying ineq. (27,4). But, as all the zeros of $f(z)$ in $|z| \leqq 1$ satisfy ineq. (27,4) also, we have fully established Th. (27,2).

Ths. (27,1) and (27,2) may be regarded as providing upper bounds for all the zeros of a given polynomial including the zero of largest modulus. To obtain a lower bound for the zero of largest modulus, we may use Grace's Theorem. Since r is a root of eq. (27,2),

$$(27,5) \qquad |a_0| + |a_1|r + \cdots + |a_{n-1}|r^{n-1} - |a_n|r^n = 0.$$

In this equation the zeros of $f(z)$ enter linearly and symmetrically. Furthermore these zeros all lie in the circle $|z| \leqq |z_1|$, where z_1 is the zero of $f(z)$ of maximum modulus. If these zeros are made to coincide at a suitably chosen point ζ in the circle $|z| \leqq |z_1|$, the equation resulting from (27,5)

$$(27,6) \qquad \begin{aligned} &|\zeta|^n + C(n,1)|\zeta|^{n-1}r + C(n,2)|\zeta|^{n-2}r^2 + \cdots \\ &\qquad + C(n,1)|\zeta|r^{n-1} - r^n = 0 \end{aligned}$$

is satisfied by the r satisfying eq. (27,5). Eq. (27,6), being the same as the equation

$$(27,7) \qquad (|\zeta| + r)^n - 2r^n = 0,$$

shows that

$$(27,8) \qquad (2^{1/n} - 1)r = |\zeta| \leqq |z_1|.$$

In other words, we have established the following result due to Birkhoff [1], Cohn [1] and Berwald [3]; namely,

THEOREM (27,3). *The zero z_1 of largest modulus of $f(z) = a_0 + a_1z + \cdots + a_nz^n$, $a_n \neq 0$, satisfies the inequality*

$$r \geqq |z_1| \geqq (2^{1/n} - 1)r, \tag{27,9}$$

where r is the positive root of the equation (27,2).

The lower limit in (27,9) is attained by $f(z) = (z + 1)^n$. For, eq. (27,2) is then $(z + 1)^n - 2z^n = 0$ and $r = 1/(2^{1/n} - 1)$. The upper limit in (27,9) is obviously attained by $f(z) = (z + 1)^n - 2z^n$.

By the above reasoning we may also show that, if z_n is the zero of $f(z)$ of smallest modulus, then $|z_n| \leqq (2^{1/n} - 1)r$. (See also ex. 27,1.)

A further improvement in bound (27,9) may be developed on use of the well known Hölder inequality

$$\sum_{j=1}^{n} \alpha_j\beta_j \leqq \left(\sum_{j=0}^{n} \alpha_j^p\right)^{1/p}\left(\sum_{j=1}^{n} \beta_j^q\right)^{1/q}, \tag{27,10}$$

where $\alpha_j > 0$, $\beta_j > 0$ for all j and $p > 1$, $q > 1$ and $(1/p) + (1/q) = 1$. When applied to (27,3), ineq. (27,10) yields the results

$$\sum_{j=0}^{n-1} |a_j|\,|z|^j \leqq \left(\sum_{j=0}^{n-1} |a_j|^p\right)^{1/p}\left(\sum_{j=0}^{n-1} |z|^{jq}\right)^{1/q}, \tag{27,11}$$

$$|f(z)| \geqq |a_n|\,|z|^n\left\{1 - A_p\left(\sum_{j=0}^{n-1} \frac{1}{|z|^{(n-j)q}}\right)^{1/q}\right\}, \tag{27,12}$$

where

$$A_p = \left(\sum_{j=0}^{n-1} |a_j/a_n|^p\right)^{1/p}. \tag{27,13}$$

Since, if $|z| > 1$,

$$\sum_{j=0}^{n-1} \frac{1}{|z|^{(n-j)q}} < \sum_{j=1}^{\infty} \frac{1}{|z|^{jq}} = \frac{1}{|z|^q - 1}, \tag{27,14}$$

we learn from (27,12) that

$$|f(z)| \geqq |a_n|\,|z^n|\left\{1 - \frac{A_p}{(|z|^q - 1)^{1/q}}\right\} > 0 \tag{27,15}$$

provided $|z|^q - 1 \geqq (A_p)^q$; i.e.,

$$|z| \geqq [1 + (A_p)^q]^{1/q}. \tag{27,16}$$

The relations (27,15), (27,16) and (27,13) lead thus to the result of Kuniyeda [1], Montel [2] and Tôya [1], which we state as

THEOREM (27,4). *For any p and q such that*

$$p > 1, \qquad q > 1, \qquad (1/p) + (1/q) = 1, \tag{27,17}$$

the polynomial $f(z) = a_0 + a_1 z + \cdots + a_n z^n$, $a_n \neq 0$, *has all its zeros in the circle*

$$|z| < \left\{1 + \left[\sum_{j=0}^{n-1} |a_j|^p / |a_n|^p\right]^{q/p}\right\}^{1/q} < (1 + n^{q/p} M^q)^{1/q} \tag{27,18}$$

where $M = \max |a_j/a_n|$, $j = 0, 1, \cdots, n-1$.

Thus, if $p = q = 2$, ineq. (27,18) becomes

$$|z| < \left\{1 + \sum_{j=0}^{n-1} |a_j|^2 / |a_n|^2\right\}^{1/2}, \tag{27,19}$$

the bound derived in Carmichael-Mason [1], Kellcher [1] and Fujiwara [1].

EXERCISES. Prove the following.

1. The zero of smallest modulus of $f(z) = a_0 + a_1 z + \cdots + a_n z^n$, $a_0 \neq 0$, lies in the ring $R \leqq |z| \leqq R/(2^{1/n} - 1)$, where R is the positive root of the equation

$$|a_0| - |a_1| z - |a_2| z^2 - \cdots - |a_n| z^n = 0.$$

Hint: Apply Th. (27,1) to $F(z) = z^n f(1/z)$.

2. All the zeros of the $f(z)$ of ex. (27,1) lie on or outside the circle

$$|z| = \min [|a_0|/(|a_0| + |a_k|)], \qquad k = 1, 2, \cdots, n.$$

3. As $p \to \infty$, the right side of (27,18) approaches the limit $1 + \max |a_j|/|a_n|$, $j = 0, 1, \cdots, n-1$, and thus Th. (27,2) is a limiting case of Th. (27,4).

4. All the zeros of $f(z) = a_0 + a_1 z + \cdots + a_n z^n$, $a_n \neq 0$, lie in the circle

$$|z| \leqq \left[1 + \left|\frac{a_0}{a_n}\right|^2 + \left|\frac{a_1 - a_0}{a_n}\right|^2 + \cdots + \left|\frac{a_n - a_{n-1}}{a_n}\right|^2\right]^{1/2}. \tag{27,20}$$

Hint: Apply (27,19) to $F(z) = (1 - z)f(z)$ [Williams 1].

5. All the zeros of $f(z) = a_0 + a_1 z + \cdots + a_n z^n$, $a_n \neq 0$, lie in the circle

$$|z| \leqq \sum_{j=1}^{n} |a_{n-j}/a_n|^{1/j}. \tag{27,21}$$

Hint: Apply ex. (17,1) successively to the polynomials $P_k(z) = a_n z^{n-k} + a_{n-1} z^{n-k-1} + \cdots + a_{n-k-1} z$, with $k = n-1, n-2, \cdots, 0$ and with $-c = a_{n-k}$ [Walsh 7].

6. All the zeros of the polynomial $z^n + a_1 z^{n-1} + \cdots + a_n$ lie in the circle $|z + ma_1| \leqq |(1 - m)a_1| + \sum_{j=2}^{n} |a_j|^{1/j}$, where m is an arbitrary real or complex constant.

7. Let $M = \max |z_j|$ of the zeros z_j ($j = 1, 2, \cdots, n$) of $f(z) = z^n + a_1 z^{n-1} + \cdots + a_n$. Then

$$M \geqq (1/n) \sum_{j=1}^{n} |a_j / C(n, j)|^{1/j}.$$

Hint: Add the n relations ($j = 1, 2, \cdots, n$)

$$| C(n, j)^{-1}a_j |^{1/j} = | C(n, j)^{-1} \sum z_1 z_2 \cdots z_j |^{1/j} \leqq M \quad \text{[Throumolopoulos 1].}$$

8. Let $f(z) = \sum_0^n a_k z^k$ and $g(z) = \sum_1^\infty b_k z^k$ with $b_k > 0$ for all k. Let r_0 be the positive root of the equation $Mg(r) = | a_0 |$, where $M = \max | a_k/b_k |$, $k = 1, 2, \cdots, n$. Then all the zeros of $f(z)$ lie in $| z | \geqq r_0$. Hint: For any zero $z = re^{i\theta}$ of $f(z)$,

$$| a_0 | g(r)^{-1} \leqq G = \left[\sum_1^n | a_k/b_k | b_k r^k \right] \Big/ \left[\sum_1^n b_k r^k \right] \leqq M$$

since G is a mean value of the quantities $| a_k/b_k |$ for $k = 1, 2, \cdots, n$ [Markovitch 3].

9. For any given positive t, let $M = \max | a_k | t^k$, $k = 1, 2, \cdots, n$. Then all the zeros of $f(z) = \sum_0^n a_k z^k$ lie in $| z | \geqq | a_0 | t/(| a_0 | + M)$. Hint: Choose $b_k = t^{-k}$ in ex. (27,8) [Landau 4; Markovitch 3].

28. **The p zeros of smallest modulus.** An important generalization of Cauchy's Th. (27,1) is the one published in 1881 by Pellet [1].

PELLET'S THEOREM. (Th. 28,1). *If for a polynomial*

$$f(z) = a_0 + a_1 z + \cdots + a_p z^p + \cdots + a_n z^n, \qquad a_p \neq 0, \tag{28,1}$$

the equation

$$F_p(z) \equiv | a_0 | + | a_1 | z + \cdots + | a_{p-1} | z^{p-1} - | a_p | z^p + | a_{p+1} | z^{p+1} + \cdots + | a_n | z^n \tag{28,2}$$

has two positive zeros r and R, $r < R$, then $f(z)$ has exactly p zeros in or on the circle $| z | \leqq r$ and no zeros in the annular ring $r < | z | < R$.

Our proof, like Pellet's, will be based upon Rouché's Theorem (Th. 1,3). Let us take a positive number ρ, $r < \rho < R$. In view of the facts that sg $F_p(z) =$ sg $F_p(0) = 1$ for $0 < z < r$ and sg $F_p(z) =$ sg $F_p(+\infty) = 1$ for $R < z < \infty$, it follows that for ϵ a sufficiently small positive number

$$F_p(\rho) < 0, \qquad r + \epsilon \leqq \rho \leqq R - \epsilon. \tag{28,3}$$

This means according to eq. (28,2) that

$$| a_p | \rho^p > \sum_{j=0}^{p-1} | a_j | \rho^j + \sum_{j=p+1}^{n} | a_j | \rho^j. \tag{28,4}$$

At this point we shall apply Rouché's Theorem to the polynomials

$$P(z) = \sum_{j=0, j \neq p}^{n} a_j z^j, \qquad Q(z) = a_p z^p. \tag{28,5}$$

Since, on the circle $|z| = \rho$, we have from (28,5) and (28,4)

$$|P(z)| \leqq \sum_{j=0, j \neq p}^{n} |a_j| \rho^j < |a_p| \rho^p = |Q(z)| \neq 0,$$

our conclusion is that, in the circle $|z| < \rho$, $f(z) = P(z) + Q(z)$ has the same number p of zeros as does $Q(z)$. Since ρ is an arbitrary number such that $r < \rho < R$, it follows that there are precisely p zeros in the region $|z| \leqq r$ and no zeros in the region $r < |z| < R$.

Pellet's Theorem, the proof of which we have just completed, may be supplemented by two theorems due to Walsh [10]. The first concerns the case that, instead of the distinct zeros r and R, $F_p(z)$ has a real double zero r while the second is a converse of Pellet's Theorem.

THEOREM (28,2). *If $F_p(z)$ has a double positive zero r, then $f(z)$ has δ ($\delta \geqq 0$) double zeros on the circle $|z| = r$, $p - \delta$ zeros inside and $n - p - \delta$ zeros outside this circle.*

THEOREM (28,3). *Let $a_0, a_1, \cdots, a_n$ be fixed coefficients and $\epsilon_0, \epsilon_1, \cdots, \epsilon_n$ be arbitrary numbers with $|\epsilon_0| = |\epsilon_1| = \cdots = |\epsilon_n| = 1$. Let ρ be any positive real number with the two properties:*

(1) *ρ is not a zero of any polynomial*

$$\phi(z) = a_0\epsilon_0 + a_1\epsilon_1 z + \cdots + a_n\epsilon_n z^n;$$

(2) *every polynomial $\phi(z)$ has p zeros ($0 < p < n$) in the circle $|z| = \rho$. Then $F_p(z)$ has two positive zeros r and R, $r < R$, and $r < \rho < R$.*

For the proof of Th. (28,2), the reader is referred to Walsh [10], but for the proof of Th. (28,3) the reader should also consult Ostrowski [2].

EXERCISES. Prove the following.

1. Th. (27,1) is the limiting case of Th. (28,1) in which all a_j, $p < j \leqq n$, are allowed to approach zero.

2. If $|a_p| > |a_0| + |a_1| + \cdots + |a_{p-1}| + |a_{p+1}| + \cdots + |a_n|$, then $f(z)$, defined by eq. (28,1), has exactly p zeros in the unit circle [Cohn 1].

29. Refinement of the bounds. In secs. 27 and 28, we took into consideration only the moduli of the coefficients of $f(z)$ in constructing some bounds for the zeros of $f(z)$. We shall now try to sharpen those bounds by taking into account also the arguments of the coefficients.

Let us divide the plane into $2p$ equal sectors S_k having their common vertex at the origin and having the rays

$$\theta = (\alpha_0 + k\pi)/p, \qquad k = 1, 2, \cdots, 2p,$$

as their bisectors. Let us denote by $G(r_0, r; p, \alpha_0)$ the boundary of the gear-wheel shaped region formed by adding to the circular region $|z| < r_0$ those

points of the annulus $r_0 \leqq |z| \leqq r$ which lie in the odd numbered sectors S_1, $S_3, \cdots, S_{2p-1}$. (See fig. 29,1.)

Following Lipka [6] in the case $p = n$ and Marden [15] in the general case, we now propose to establish a refinement of Pellet's Theorem (Th. 28,1).

THEOREM (29,1). *If the polynomial*

$$f(z) = a_0 + a_1 z + \cdots + a_p z^p + \cdots + a_n z^n \tag{29,1}$$

with

$$a_0 a_1 a_p a_n \neq 0 \qquad \textit{and} \qquad \alpha_0 = \arg (a_0/a_p) \tag{29,2}$$

be such that the equation

$$\begin{aligned} F_p(z) \equiv |a_0| + |a_1| z + \cdots + |a_{p-1}| z^{p-1} - |a_p| z^p \\ + |a_{p+1}| z^{p+1} + \cdots + |a_n| z^n = 0 \end{aligned} \tag{29,3}$$

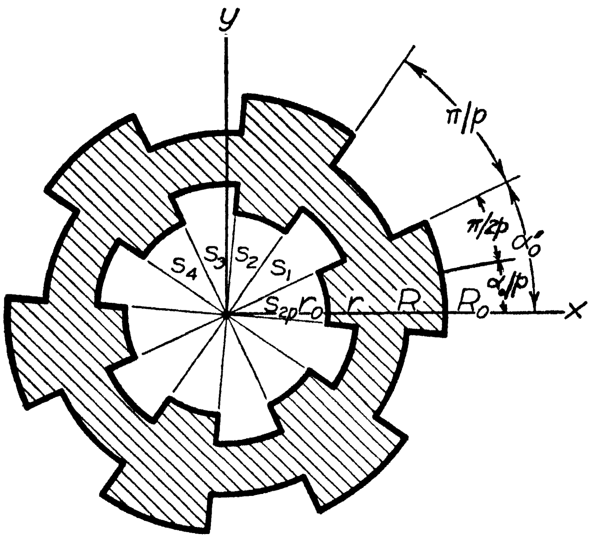

FIG. (29, 1)

has two positive zeros r and R, r < R, then the equation

$$\begin{aligned} \Phi_p(z) \equiv |a_1| + |a_2| z + \cdots + |a_{p-1}| z^{p-2} - |a_p| z^{p-1} \\ + |a_{p+1}| z^p + \cdots + |a_n| z^{n-1} = 0 \end{aligned} \tag{29,4}$$

has two positive zeros r_0 and R_0 with $r_0 < r < R < R_0$. Furthermore, the polynomial $f(z)$ has precisely p zeros in or on the curve $G(r_0, r; p, \alpha_0)$ and no zeros in the annular region between the curves $G(r_0, r; p, \alpha_0)$ and $G(R, R_0; p, \alpha_0 + \pi)$.

As to the existence of the roots r_0 and R_0, let us note that, according to (29,3) and (29,4),

(29,5) $$F_p(z) = |a_0| + z\Phi_p(z).$$

Thus,

(29,6) $$\Phi_p(r) = -|a_0|/r, \qquad \Phi_p(R) = -|a_0|/R.$$

Since

$$\Phi_p(0) = |a_1| > 0 \text{ and } \Phi_p(+\infty) > 0,$$

it follows that $\Phi_p(z) = 0$ has two roots r_0 and R_0 such that $0 < r_0 < r < R < R_0$ and that, for $\epsilon > 0$ and sufficiently small,

(29,7) $$\Phi_p(\rho) < 0 \qquad \text{for} \qquad r_0 + \epsilon \leqq \rho \leqq R_0 - \epsilon.$$

Let us now set $z = \rho e^{i\theta}$ and

(29,8) $$a_k/a_p = A_k e^{\alpha_k i}, \qquad A_k \geqq 0, \qquad k = 0, 1, 2, \cdots, n.$$

For the real part of $[\rho^p f(z)/a_p z^p]$, we then have

(29,9) $$\Re[\rho^p f(z)/a_p z^p] = \sum_{j=0, j\neq p}^{n} A_j\rho^j \cos[(p-j)\theta - \alpha_j] + \rho^p.$$

On the other hand, inequalities (28,4) and (29,7) may be written as

(29,10) $$\rho^p > \sum_{j=0, j\neq p}^{n} A_j\rho^j, \qquad r + \epsilon \leqq \rho \leqq R - \epsilon,$$

(29,11) $$\rho^p > \sum_{j=1, j\neq p}^{n} A_j\rho^j, \qquad r_0 + \epsilon \leqq \rho \leqq R_0 - \epsilon.$$

Substituting these into (29,9), we have

(29,12) $$\Re[\rho^p f(z)/a_p z^p] > \sum_{j=0, j\neq p}^{n} A_j\rho^j\delta_j, \qquad r + \epsilon \leqq \rho \leqq R - \epsilon,$$

(29,13) $$\Re[\rho^p f(z)/a_p z^p] > A_0 \cos[p\theta - \alpha_0] + \sum_{j=1, j\neq p} A_j\rho^j\delta_j, \qquad r_0 + \epsilon \leqq \rho \leqq R_0 - \epsilon,$$

where $\delta_j = 1 + \cos[(p-j)\theta - \alpha_j]$. It is clear that the right side of inequality (29,12) is non-negative for all angles θ and that the right side of inequality (29,13) is non-negative for angles θ in the ranges

$$-\pi/2 + 2\pi k \leqq p\theta - \alpha_0 \leqq \pi/2 + 2\pi k, \qquad k = 0, 1, \cdots, p-1;$$

that is, in the ranges

(29,14) $$|\theta - (\alpha_0 + 2k\pi)/p| \leqq \pi/2p, \qquad k = 1, 2, \cdots, p,$$

constituting the even numbered sectors S_{2k}.

Furthermore, we see that $f(z)$ has no zeros on the rays

$$\theta = [2\alpha_0 + (4k + 1)\pi]/2p, \qquad k = 0, 1, \cdots, p - 1,$$

inside the annular region $r_0 < |z| < R_0$.

Let us now apply ex. (1,9), taking as C the curve $G(r_0 + \epsilon_1, r + \epsilon_2; p, \alpha_0)$ where $0 < \epsilon_1 < R_0 - r_0$ and $0 < \epsilon_2 < R - r$. Due to the fact that $\Re[\rho^p f(z)/a_p z^p] > 0$ along this curve for any of the above values of ϵ_1 and ϵ_2, we may infer that $f(z)$ has the same number p of zeros as $a_p z^p$ inside the curve $G(r_0, r; p, \alpha_0)$ and no zeros between curves $G(r_0, r; p, \alpha_0)$ and $G(R, R_0; p, \alpha_0 + \pi)$.

Incidentally, if in ex. (1,9) we take as C any circle $|z| = \rho$, $r < \rho < R$, we obtain another proof of Pellet's Theorem, since $\Re[\rho^p f(z)/a_p z^p] > 0$ along this circle.

EXERCISES. Prove the following.

1. All the zeros of $f(z) = a_0 + a_1 z + \cdots + a_n z^n$, $a_n \neq 0$, lie in the gear-wheel region $G(r_0, r; n, \alpha_0)$ where $\alpha_0 = \arg(a_0/a_n)$, where r is the positive root of eq. (27,2) and r_0 the positive root of the equation

$$|a_1| + |a_2| z + \cdots + |a_{n-1}| z^{n-2} - |a_n| z^{n-1} = 0 \qquad \text{[Lipka 6].}$$

2. If the polynomial $f(z) = \sum_0^n a_k z^k$ with $a_k a_p \neq 0$ and $\arg a_k/a_p = \alpha_k$ is such that $F_p(z)$ has two positive zeros r and R, $r < R$, then the polynomial

$$\Psi_k(z) = F_p(z) - |a_k| z^k, \qquad k \neq p,$$

has two positive zeros r_k and R_k with $r_k < r < R < R_k$, and the polynomial $f(z)$ has precisely p zeros in or on the curve $G(r_k, r; p - k, \alpha_k)$ and no zeros between the curves $G(r_k, r; p - k, \alpha_k)$ and $G(R, R_k; p - k, \alpha_k + \pi)$ [Marden 15].

3. If the power series $f(z) = \sum_0^\infty a_j z^j$ with $a_k a_p \neq 0$, $\arg a_k/a_p = \alpha_k$, and with a radius of convergence $\rho > 0$ is such that each polynomial

$$F_p^{(n)}(z) = |a_0| + |a_1| z + \cdots + |a_{p-1}| z^{p-1} - |a_p| z^p + \cdots + |a_{p+1}| z^{p+1} + \cdots + |a_n| z^n$$

has a positive zero $r^{(n)} < \rho$, then the function $F_p(z) = \lim_{n\to\infty} F_p^{(n)}(z)$ has a positive zero $r < \rho$; the function

$$\Psi_k(z) = F_p(z) - |a_k| z^k, \qquad k \neq p,$$

has a positive zero $r_k < r$, and the function $f(z)$ has p zeros in or on the curve $G(r_k, r; p - k, \alpha_k)$ and, hence, in the curve $G(r_k, \rho; p - k, \alpha_k)$ [Marden 15].

30. Applications. As a first application of Th. (29,1), we shall establish a result due to Marden [15].

THEOREM (30,1). *Let*

$$f(z) = b_0 e^{i\beta_0} + (b_1 - b_0) e^{i\beta_1} z + \cdots + (b_{n-1} - b_{n-2}) e^{i\beta_{n-1}} z^{n-1} - b_{n-1} e^{i\beta_n} z^n, \tag{30,1}$$

where $b_{p-1} \leqq b_{p-2} \leqq \cdots \leqq b_0 \leqq 0 < b_{n-1} \leqq b_{n-2} \leqq \cdots \leqq b_p$. *Let* $\beta_0' = \beta_0 - \beta_p - \pi$ *and let*

$$g(z) = b_0 + b_1 z + \cdots + b_{n-1} z^{n-1}. \tag{30,2}$$

Let r_0 *be the smaller positive root of the equation*

$$\Phi_p(z) \equiv (b_0 - b_1) + (b_1 - b_2)z + \cdots + (b_{n-2} - b_{n-1})z^{n-2} + b_{n-1}z^{n-1} = 0. \tag{30,3}$$

Then, if $g(1) > 0$, $f(z)$ *has exactly* p *zeros in the curve* $G(r_0, 1; p, \beta_0')$ *and* $g(z)$ *has* p *zeros in the curve* $G(r_0, 1; p, \pi)$. *If* $g(1) < 0$, $f(z)$ *has exactly* p *zeros in or on the curve* $G(r_0, 1; p, \beta_0')$ *and* $g(z)$ *has* $p - 1$ *zeros in or on the curve* $G(r_0, 1; p, \pi)$.

Insofar as it concerns the zeros of $g(z)$, Th. (30,1) reduces to a result due to Berwald [2] when the curve $G(r_0, 1; p, \pi)$ is replaced by the circle $|z| = 1$. Thus Th. (30,1) is a refinement of Berwald's result.

To prove this theorem, we make use of the fact that corresponding to the $f(z)$ in (30,1), the polynomial (29,3) is

$$F_p(z) = -b_0 + (b_0 - b_1)z + \cdots + (b_{p-2} - b_{p-1})z^{p-1} - (b_p - b_{p-1})z^p + (b_p - b_{p+1})z^{p+1} + \cdots + (b_{n-2} - b_{n-1})z^{n-1} + b_{n-1}z^n. \tag{30,4}$$

This function may also be written as

$$F_p(z) = (z - 1)g(z). \tag{30,5}$$

Clearly $F_p(1) = 0$. Since $F_p(1 + \delta) = \delta g(1 + \delta)$, $F_p(z)$ changes from $-$ to $+$ or from $+$ to $-$ at $z = 1$ according as $g(1) > 0$ or $g(1) < 0$. In the notation of Th. (29,1),

$$r_0 < r < 1 = R < R_0 \qquad \text{if } g(1) > 0; \tag{30,6}$$

$$r_0 < r = 1 < R < R_0 \qquad \text{if } g(1) < 0; \tag{30,7}$$

$$\alpha_0 = \beta_0 - \beta_p - \pi = \beta_0'. \tag{30,8}$$

Since $f(z)$ has p zeros in or on the curve $G(r_0, r; p, \beta_0')$ according to Th. (29,1), it has p zeros in $G(r_0, 1; p, \beta_0')$ if $g(1) > 0$ and p zeros in or on $G(r_0, 1; p, \beta_0')$ if $g(1) < 0$. This proves Th. (30,1) as far as $f(z)$ is concerned.

To prove Th. (30,1) with respect to $g(z)$, we need merely note that the zeros of $g(z)$ are those of $F_p(z)$ except for $z = 1$ and that, considered as a special case of (30,1), $F_p(z)$ has its $\beta_0 = \pi$ and $\beta_p = -\pi$ and thus its $\alpha_0 = \pi$.

As our second application of Th. (29,1), we shall establish a result somewhat more general than the one given in Marden [15].

THEOREM (30,2). *Let* $\lambda_1, \lambda_2, \cdots, \lambda_q$ *and* $\mu_1, \mu_2, \cdots, \mu_{q-1}$ *be any two sets of positive numbers such that*

$$\sum_{j=1}^{q} (1/\lambda_j) = 1, \qquad \sum_{j=1}^{q-1} (1/\mu_j) = 1; \qquad \mu_j \leqq \lambda_j, \qquad j = 1, 2, \cdots, q - 1.$$

For the polynomial

$$f(z) = a_0 + a_1 z^{n_1} + a_2 z^{n_2} + \cdots + a_q z^{n_q}, \tag{30,9}$$

where

$$a_0 a_q \neq 0 \quad \text{and} \quad 0 = n_0 < n_1 < n_2 < \cdots < n_q = n,$$

let

$$M = \max [\lambda_k \mid a_k \mid / \mid a_q \mid]^{1/(n-n_k)}, \qquad k = 0, 1, \cdots, q-1; \tag{30,10}$$

$$M_0 = \max [\mu_k \mid a_k \mid / \mid a_q \mid]^{1/(n-n_k)}, \qquad k = 1, 2, \cdots, q-1. \tag{30,11}$$

Then all the zeros of $f(z)$ *lie in or on the gear-wheel curve* $G(M_0, M; n, \alpha_0)$ *where* $\alpha_0 = \arg (a_0/a_q)$.

Proof. From eqs. (30,10) and (30,11), it follows that $0 < M_0 \leqq M$ and also that

$$\lambda_k \mid a_k \mid \leqq \mid a_q \mid M^{n-n_k}, \qquad \mu_k \mid a_k \mid \leqq \mid a_q \mid M_0^{n-n_k}.$$

Hence,

$$\sum_{k=0}^{q-1} \mid a_k \mid M^{n_k} \leqq \sum_{k=0}^{q-1} (1/\lambda_k) \mid a_q \mid M^n = \mid a_q \mid M^n, \tag{30,12}$$

$$\sum_{k=1}^{q-1} \mid a_k \mid M_0^{n_k} \leqq \sum_{k=1}^{q-1} (1/\mu_k) \mid a_q \mid M_0^n = \mid a_q \mid M_0^n . \tag{30,13}$$

From an equality in (30,12), we would infer that M is the positive root r of the equation

$$\mid a_0 \mid + \mid a_1 \mid z^{n_1} + \cdots + \mid a_{q-1} \mid z_n^{n_{q-1}} - \mid a_q \mid z^n = 0, \tag{30,14}$$

whereas from an inequality in (30,12), we would infer that $M > r$. Similarly, from an equality in (30,13) we would infer that M_0 is the positive root r_0 of the equation

$$\mid a_1 \mid z^{n_1} + \mid a_2 \mid z^{n_2} + \cdots + \mid a_{q-1} \mid z^{n_{q-1}} - \mid a_q \mid z^n = 0, \tag{30,15}$$

whereas from an inequality in (30,13) we would infer that $M_0 > r_0$. Since we recognize eqs. (30,14) and (30,15) to be respectively $F_n(z) = 0$ and $\Phi_n(z) = 0$, we conclude from Th. (29,1) that all the zeros of $f(z)$ lie in or on $G(r_0, r; n, \alpha_0)$ and therefore in or on $G(M_0, M; n, \alpha_0)$, thus establishing Th. (30,2).

If in (30,9) each $n_k = 1 + k$ and if the curve $G(M_0, M; n, \alpha_0)$ is replaced by the circle $\mid z \mid = M$, then Th. (30,2) reduces to a result of Fujiwara [3]. Thus Th. (30,2) is both a generalization and refinement of Fujiwara's result.

Of special interest, are the following two sets of the λ_j and the μ_j :

$$\begin{cases} \lambda_j = q, & j = 0, 1, \cdots, q-1; \\ \mu_k = q - 1, & k = 1, 2, \cdots, q-1; \end{cases} \tag{30,16}$$

$$(30,17)\qquad \begin{cases} \lambda_j = \sum_{\nu=0}^{q-1} |a_\nu|/|a_j|, & j = 0, 1, \cdots, q-1; \\ \mu_k = \sum_{\nu=1}^{q-1} |a_\nu|/|a_k|, & k = 1, 2, \cdots, q-1. \end{cases}$$

On use of the set (30,16), we deduce at once from Th. (30,2) the following result.

COROLLARY (30,2a). *For the polynomial $f(z)$ in eq.* (30,9), *let*

$$M = \max\,[q\,|a_k|/|a_q|]^{1/(n-n_k)}, \qquad k = 0, 1, \cdots, q-1$$

and

$$M_0 = \max\,[(q-1)\,|a_k|/|a_q|]^{1/(n-n_k)}, \qquad k = 1, 2, \cdots, q-1.$$

Then all the zeros of $f(z)$ lie in or on the curve $G(M_0, M; n, \alpha_0)$ where $\alpha_0 = \arg(a_0/a_q)$.

On use of the set (30,17), we see on setting

$$(30,18)\qquad \rho = \sum_{j=0}^{q-1} |a_j|/|a_q|, \qquad \rho_0 = \sum_{j=1}^{q-1} |a_j|/|a_q|,$$

that

$$(30,19)\qquad \begin{aligned} M &= \max \rho^{1/(n-n_k)} = \max(\rho, \rho^{1/n}), \\ M_0 &= \max \rho_0^{1/(n-n_k)} = \max(\rho_0, \rho_0^{1/(n-n_1)}). \end{aligned}$$

We thereby derive

COROLLARY (30,2b). *For the polynomial $f(z)$ of eq.* (30,9), *let ρ and ρ_0 be computed from eqs.* (30,18) *and let*

$$\kappa = \max(\rho, \rho^{1/n}), \qquad \kappa_0 = \max(\rho_0, \rho_0^{1/(n-n_1)}).$$

Then all the zeros of $f(z)$ lie in or on the curve $G(\kappa_0, \kappa; n, \alpha_0)$ where $\alpha_0 = \arg a_0/a_q$.

Various other corollaries may be deduced from Th. (30,2) on making other special choices of the λ_j and μ_j, as will be seen in the exercises below.

EXERCISES. Prove the following.

1. *Eneström-Kakeya Theorem.* If all the a_j are real and if $a_0 \geqq a_1 \geqq \cdots \geqq a_n > 0$, then $f_n(z) = \sum_0^n a_k z^k \neq 0$ for $|z| < 1$ [Eneström 1, Kakeya 1, Hurwitz 3]. Hint: Use Th. (30,1). Alternatively, for a fixed r, $0 < r < 1$, construct the polygonal line P joining in succession the points $Z_0 = 0$, $Z_k = f_k(re^{i\theta})$, $k = 1, 2, \cdots, n$. Show, if $S_k = S(\text{seg } Z_k Z_{k+1}, \theta/2)$, that $S_k \subseteq S_{k-1}$ for all $k \geqq 1$ and that no S_k, $k > 0$, contains $z = 0$ [Tomić 1].

2. All the zeros of the polynomial $f(z) = a_0 + a_1 z + \cdots + a_n z^n$ having real positive coefficients a_j lie in the circle $|z| \leqq \rho$ where $\rho = \max\,[a_k/a_{k+1}]$, $k = 0, 1, \cdots, n-1$. Hint: Apply ex. 1 to $g(z) = f(\rho z)$.

3. The real polynomial

$$h(z) = a_0 + a_1 z + \cdots + a_k z^k - a_{k+1} z^{k+1} - \cdots - a_n z^n, \qquad a_j > 0, \text{ all } j,$$

has no non-real zeros in the annular ring $\rho_1 < |z| < \rho_2$ where $\rho_1 = \max (a_j/a_{j+1})$, $j = 0, 1, 2, \cdots, k - 1$, and $\rho_2 = \max (a_j/a_{j+1})$, $j = k, k - 1, \cdots, n - 1$. How many zeros does $f(z)$ have in the circle $|z| < \rho_1$ [Hayashi 2, Hurwitz 3]?

4. All the zeros of $f(z) = a_0 + a_1 z + \cdots + a_n z^n$ lie in or on the curve $G(M_0, M; n, \alpha_0)$ where $\alpha_0 = \arg (a_0/a_n)$,

$$M = \max \rho \, |a_{n-k}/a_n|^{1/k}, \qquad k = 1, 2, \cdots, n,$$

$$M_0 = \max \rho_0 \, |a_{n-k}/a_n|^{1/k}, \quad k = 1, 2, \cdots, n - 1,$$

and ρ and ρ_0 are the positive roots of the equations

$$\rho^n = \rho^{n-1} + \rho^{n-2} + \cdots + 1, \qquad \rho_0^{n-1} = \rho_0^{n-2} + \rho_0^{n-3} + \cdots + 1.$$

Hint: Choose $\lambda_k = \rho^k$ and $\mu_k = \rho_0^k$ and apply Th. (30,2).

5. All the zeros of $f(z) = a_0 + a_1 z + \cdots + a_n z^n$ lie in or on the curve $G(M_0, M; n, \alpha_0)$ where $\alpha_0 = \arg (a_0/a_n)$ and where

$$M = 2 \max \{|a_{n-1}/a_n|, |a_{n-2}/a_n|^{1/2}, \cdots, |a_1/a_n|^{1/(n-1)}, |a_0/2a_n|^{1/n}\},$$

$$M_0 = 2 \max \{|a_{n-1}/a_n|, |a_{n-2}/a_n|^{1/2}, \cdots, |a_2/a_n|^{1/(n-2)}, |a_1/2a_n|^{1/(n-1)}\}.$$

Hint: In Th. (30,2), choose $\lambda_k = 2^k$, $k = 1, 2, \cdots, n - 1$; $\lambda_n = 2^{n-1}$; $\mu_k = 2^k$, $k = 1, 2, \cdots, n - 2$; $\mu_{n-1} = 2^{n-2}$.

6. All the zeros of $f(z) = a_0 + a_1 z + \cdots + a_n z^n$ lie in the circle $|z| \leqq r$, $r = \max (|a_0|/|a_1|, 2|a_k/a_{k+1}|)$, $k = 1, 2, \cdots, n - 1$. Hint: Show that $|a_{k+1}| r^{k+1} \geqq 2 |a_k| r^k$ for $k \geqq 1$; $|a_1| r \geqq |a_0|$ and thus $|a_n r^n| \geqq |a_0| + \cdots + |a_{n-1}| r^{n-1}$. Remark: The limit is attained by $f(z) = 2 + z + z^2 + \cdots + z^{n-1} - z^n$ [Kojima 1, 2].

7. Let λ_j be positive numbers such that $\sum_{j=1}^{n} (1/\lambda_j) = 1$. If there exists an $r > 0$ such that

$$\max [\lambda_j \, |a_{p-j}/a_p|]^{1/j} \leqq r \leqq \min [\lambda_{k+p}^{-1} \, |a_p/a_{p+k}|]^{1/k},$$

for $j = 1, 2, \cdots, p$ and $k = 1, 2, \cdots, n - p$, then there are p zeros of $f(z)$ in $|z| \leqq r$.

8. All the zeros of the polynomial $f(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + z^n$ lie in the circle $|z| \leqq \max (L, L^{1/(n+1)})$, where L is the length of the polygonal line joining in succession the points $0, a_0, a_1, \cdots, a_{n-1}, 1$. Hint: Apply Cor. (30,2b) to $g(z) = (1 - z)f(z)$. [Montel 2, Marty 1].

9. If $a_j > 0$, $j = 0, 1, \cdots, n$, $a_{-1} = a_{n+1} = 0$, and if ρ_1 and ρ_2 can be found ($\rho_2 \geqq \rho_1 > 0$) such that $\rho_1 \rho_2 a_{j+1} - (\rho_1 + \rho_2) a_j + a_{j-1} > 0$ for $j = 0, 1, \cdots, p - 1, p + 1, \cdots, n$, then p zeros of $f(z)$ lie in $|z| < \rho_1$ and $n - p$ lie in $|z| > \rho_2$. Hint: Apply Th. (28,1) to $(\rho_2 - z)(\rho_1 - z)f(z)$ [Egerváry 4].

10. The real polynomial $f(z) = a_0 + a_1 z + \cdots + a_n z^n$, $a_0 \geqq a_1 \geqq \cdots \geqq a_n$,

has a non-real zero z_1 of modulus one if and only if the a_j fall into groups of m successive equal coefficients; that is, defining $a_j = 0$ for $j > n$, we have

$$(30,20) \qquad a_0 = a_1 = \cdots = a_{m-1} > a_m = a_{m+1} = \cdots = a_{2m-1} \\ > a_{2m} = a_{2m+1} = \cdots .$$

Hint: Obviously, $z_1 \neq 1$ and $g(z_1) = (1 - z_1)f(z_1) = 0$;

$$a_0 = \left| \sum_1^{n+1} (a_{k-1} - a_k)z_1^k \right| < \sum_1^{n+1} (a_{k-1} - a_k) = a_0 ,$$

unless all the terms $(a_{k-1} - a_k)z_1^k$ are real and positive. Let m be the least number for which $z_1^m = 1$. For the converse, note that eq. (30,20) implies that $1 + z + z^2 + \cdots + z^{m-1}$ is a factor of $f(z)$ [Hurwitz 3]. Alternatively, show for $r = 1$ the polygonal line P of ex. (30,1) must reduce to one or more regular polygons if $z = e^{i\theta}$ is to be a zero of $f(z)$ [Tomić 1].

CHAPTER VIII

BOUNDS FOR p ZEROS AS FUNCTIONS OF $p + 1$ COEFFICIENTS

31. **Existence of such bounds.** In the preceding chapter we obtained several bounds which were valid either for all the zeros or for p, $p < n$, of the zeros of the polynomial

$$(31,1) \qquad f(z) = a_0 + a_1 z + \cdots + a_n z^n.$$

In either case the bounds were expressed as functions of all the coefficients. While clearly the bounds for the moduli of all n zeros should involve all $n + 1$ a_j, it is natural to ask whether there exist some bounds for the p, $p < n$, zeros of smallest modulus which would be independent of certain a_j.

This question was first raised in 1906-7 by Landau in connection with his study of the Picard Theorem. In [1] and [2] Landau proved that every trinomial

$$(31,2) \qquad a_0 + a_1 z + a_n z^n, \qquad a_1 \neq 0, \qquad n \geqq 2,$$

has at least one zero in the circle $|z| \leqq 2 |a_0/a_1|$ and that every quadrinomial

$$(31,3) \qquad a_0 + a_1 z + a_m z^m + a_n z^n, \qquad a_1 \neq 0, \qquad 2 \leqq m < n,$$

has at least one zero in the circle $|z| \leqq (17/3) |a_0/a_1|$. These two polynomials are of the lacunary type

$$(31,4) \qquad a_0 + a_1 z + \cdots + a_p z^p + a_{n_1} z^{n_1} + \cdots + a_{n_k} z^{n_k},$$

with $a_p \neq 0$ and $1 \leqq p < n_1 < n_2 < \cdots < n_k$ which will be treated in secs. 34 and 35. In those sections we shall establish the existence of a circle $|z| = R(a_0, a_1, \cdots, a_p, k)$ which contains at least p zeros of every polynomial (31,4).

In order to gain some insight into the problem under discussion, let us first prove that if in eq. (31,1) one of the coefficients $a_0, a_1, \cdots, a_{p-1}$ is arbitrary, then at least $n - p + 1$ zeros of polynomial (31,1) may be made arbitrarily large in modulus. Let us select ρ as an arbitrary, but fixed, positive number. If an $|a_k|$, $0 \leqq k \leqq p - 1$, is arbitrary, then we may choose that $|a_k|$ so large that irrespective of the values of the other $|a_j|$, $j \neq k$,

$$|a_k| \rho^k > \sum_{j=0}^{k-1} |a_j| \rho^j + \sum_{j=k+1}^{n} |a_j| \rho^j.$$

It follows from Pellet's Theorem (Th. 28,1) that $n - k$ zeros of $f(z)$ exceed ρ in modulus. That is, at least $n - p + 1$ zeros of $f(z)$ surpass ρ in modulus.

Let us also show that, even though $a_0, a_1, \cdots, a_{p-1}$ are all fixed, $n - p + 1$ zeros of polynomial (31,1) may be made arbitrarily large if all the remaining co-

efficients a_j, $j \geqq p$, are arbitrary. This becomes clear if we consider the reciprocal polynomial

$$F(z) = z^n f(1/z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{p-1} z^{n-p+1} + a_p z^{n-p} + \cdots + a_n .$$

If for all $j \geqq p$ we choose the $|a_j|$ sufficiently small, then by Th. (1,4) the zeros of $F(z)$ may be brought as close as desired to the zeros of

$$F_0(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{p-1} z^{n-p+1}$$

and thus at least $n - p + 1$ zeros of $F(z)$ may be made to lie in an arbitrarily small circle $|z| = 1/\rho$. That is, at least $n - p + 1$ zeros of $f(z)$ may be made to lie outside an arbitrarily large circle $|z| = \rho$.

Finally, let us use the reasoning in Montel [3] to show that, if the coefficients $a_0, a_1, \cdots, a_{p-1}$ and a_{p+h} for some h, $0 \leqq h \leqq n - p$, are fixed, then p zeros of $f(z)$ are bounded. Were the contrary true, we could select a monotonically increasing sequence of positive numbers ρ_m, with $\rho_m \to \infty$ as $m \to \infty$, and corresponding to each ρ_m, we could select a polynomial

$$f_m(z) = \sum_{j=0}^{n} a_j^{(m)} z^j, \text{ having } a_j^{(m)} = a_j \text{ for } j = 0, 1, \cdots p - 1, p + h$$

and having at most $p - 1$ zeros in the circle $|z| \leqq \rho_m$. Defining A_m as max $|a_j^{(m)}|$ for $j = 0, 1, \cdots, n$, we distinguish two cases according as A_m does or does not remain bounded as $m \to \infty$. In the first case, we may select a subsequence of the $f_m(z)$ approaching uniformly as limit a polynomial $\phi(z)$ of degree at least $p + h$. In the second case, we may introduce the polynomials $g_m(z) = f_m(z)/A_m$, which have the same zeros as the $f_m(z)$ and in which for m sufficiently large the coefficient of the maximum modulus one is that of a term of degree at least p. Thus we may select a subsequence of the $g_m(z)$ approaching uniformly as limit a polynomial $\psi(z)$ of degree at least p. However, we learn from Hurwitz' Theorem (Th. 1,5) that neither $\phi(z)$ nor $\psi(z)$ can have more than $p - 1$ zeros and hence neither can have a degree greater than $p - 1$. Thus, the assumption that p zeros of $f(z)$ are not bounded has led to a contradiction and must therefore be false.

EXERCISES. Prove the following.

1. If the coefficients a_j of $f(z)$ satisfy p linear equations,

$$\lambda_{j0} a_0 + \lambda_{j1} a_1 + \cdots + \lambda_{jn} a_n = 0, \qquad j = 0, 1, \cdots, p - 1, \qquad p \leqq n,$$

with a nonvanishing determinant $|\lambda_{jk}|$, $j, k = 0, 1, \cdots, p - 1$, then $f(z)$ has p zeros in a circle $|z| = R$, where R is a function only of the λ_{jk} [Dieudonné 11, p. 22].

2. Let $f(z) = \sum_0^n a_k z^k$, $g(z) = \sum_0^n b_k z^k$ and $F(z) = f(z)/g(z)$. If $f(z)$ has p zeros in the circle $|z| \leqq R = R(a_0, a_1, \cdots, a_m)$ for fixed p and m, $0 \leqq p \leqq n$ and $0 \leqq m \leqq n$, and for arbitrary a_j, $j > m$, and if $b_k = Aa_k$ for $0 \leqq k \leqq m$, then $F(z)$ assumes every value Z at least p times in $|z| \leqq R$ [Nagy 17]. Hint: Study the zeros of $h(z) = f(z) - Zg(z)$.

32. Construction of some bounds. Our first bounds upon the p zeros of smallest modulus as functions of the first $p + 1$ coefficients will be constructed by modification of the previously developed bounds upon all n zeros of $f(z)$ as functions of all $n + 1$ coefficients a_j. The method to be used is one due to Montel [3].

Let us label the zeros α_j of an nth degree polynomial

$$f(z) = a_0 + a_1 z + \cdots + a_n z^n \tag{32,1}$$

in the order of decreasing modulus:

$$| \alpha_1 | \geqq | \alpha_2 | \geqq \cdots \geqq | \alpha_n |. \tag{32,2}$$

Then, in or on the circle $| z | \leqq r_p = | \alpha_{n-p+1} |$ lie the p smallest (in modulus) zeros α_j $(j = n - p + 1, n - p + 2, \cdots, n)$ of $f(z)$. These α_j are the zeros of the polynomial

$$f_{n-p}(z) = \frac{f(z)}{(\alpha_1 - z)(\alpha_2 - z) \cdots (\alpha_{n-p} - z)} = \sum_{j=0}^{p} a_j^{(n-p)} z^j. \tag{32,3}$$

It is to $f_{n-p}(z)$ that we now shall apply the results of the previous chapter, so as to obtain some estimates on the size of r_p.

For this purpose, we need first to derive expressions for the coefficients $a_j^{(n-p)}$ in terms of the a_j and the α_j. Let us note that for $| z | < r_p \leqq | \alpha_j |$, $j = 1, 2, \cdots, n - p$,

$$\prod_{j=1}^{n-p} \left(1 - \frac{z}{\alpha_j}\right)^{-1} = \prod_{j=1}^{n-p} \sum_{k=0}^{\infty} \left(\frac{z}{\alpha_j}\right)^k = \sum_{k=0}^{\infty} S_k z^k \tag{32,4}$$

where S_k is the sum of all possible products of total degree k formed from the quantities $(1/\alpha_j)$. Thus,

$$S_0 = 1, \qquad S_1 = \sum (1/\alpha_{j_1}),$$

$$S_2 = \sum \frac{1}{\alpha_{j_1}^2} + \sum \frac{1}{\alpha_{j_1}\alpha_{j_2}},$$

$$S_3 = \sum \frac{1}{\alpha_{j_1}^3} + \sum \frac{1}{\alpha_{j_1}\alpha_{j_2}} \left(\frac{1}{\alpha_{j_1}} + \frac{1}{\alpha_{j_2}}\right) + \sum \frac{1}{\alpha_{j_1}\alpha_{j_2}\alpha_{j_3}},$$

where $j_1 = 1, 2, \cdots, n - p$, but $j_{i+1} = j_i + 1, j_i + 2, \cdots, n - p$ for $i = 1, 2, \cdots$.

Using this notation, we may express eq. (32,3) as

$$f_{n-p}(z) = \frac{1}{\alpha_1 \alpha_2 \cdots \alpha_{n-p}} \left(\sum_{j=0}^{n} a_j z^j\right)\left(\sum_{k=0}^{\infty} S_k z^k\right). \tag{32,5}$$

Since $f_{n-p}(z)$ is a polynomial of degree p, expansion (32,5) converges to $f_{n-p}(z)$ for all z and the combined coefficient of each term z^k, $k > p$, is zero. That is to say,

(32,6) $$f_{n-p}(z) = \frac{1}{\alpha_1\alpha_2 \cdots \alpha_{n-p}} \left\{ \sum_{k=0}^{p} (a_k + a_{k-1}S_1 + \cdots + a_0 S_k) z^k \right\}.$$

The general coefficient in (32,3) according to (32,6) is

(32,7) $$a_k^{(n-p)} = \frac{1}{\alpha_1\alpha_2 \cdots \alpha_{n-p}} \sum_{j=0}^{k} a_{k-j} S_j \, .$$

For $k = p$, we obtain from eq. (32,3) the simpler formula

(32,8) $$a_p^{(n-p)} = (-1)^{n-p} a_n \, .$$

Thus we have

(32,9) $$| a_k^{(n-p)} | \leqq (r_p)^{-n+p} \sum_{j=0}^{k} | a_{k-j} | \, | S_j | .$$

In order to find a bound for $| S_k |$, let us observe that S_k is a kth degree, symmetric function of the α_j^{-1} , $j = 1, 2, \cdots, n - p$, with $| \alpha_j | > r_p$. For the values $\alpha_1 = \alpha_2 = \cdots = \alpha_{n-p} = 1$, eq. (32,4) becomes

(32,10) $$(1 - z)^{-n+p} = \sum_{0}^{\infty} C(n - p + k - 1, k) z^k.$$

Hence $C(n - p + k - 1, k)$ is the number of terms in S_k . Since each term is of modulus not greater than $1/r_p^k$,

(32,11) $$| S_k | \leqq C(n - p + k - 1, k) r_p^{-k}.$$

It now follows from ineq. (32,9) that

(32,12) $$| a_k^{(n-p)} | \leqq (r_p)^{-n+p} \sum_{0}^{k} C(n - p + j - 1, j) \, | a_{k-j} | \, r_p^{-j}.$$

As a first application of this formula, let us set

(32,13) $$M_p = \max | a_j/a_n |, \qquad j = 0, 1, \cdots, p.$$

From ineq. (32,12), we then obtain

(32,14) $$| a_k^{(n-p)} | \leqq M_p \, | a_n | \, r_p^{-n+p} \sum_{0}^{k} C(n - p + j - 1, j) r_p^{-j}.$$

If $r_p > 1$, we may replace the right side of (32,14) by a convergent infinite series which may be evaluated by setting $z = 1/r_p$ in eq. (32,10). Thus,

(32,15) $$| a_k^{(n-p)} | < M_p \, | a_n | \, r_p^{-n+p} (1 - r_p^{-1})^{-n+p}.$$

On use of eq. (32,8), we may write ineq. (32,15) as

(32,16) $$| a_k^{(n-p)} | < M_p \, | a_p^{(n-p)} | \, (r_p - 1)^{-n+p}.$$

This inequality permits the immediate application of Th. (27,2) to the polynomial

$$f_{n-p}(z) = a_0^{(n-p)} + a_1^{(n-p)} z + \cdots + a_p^{(n-p)} z^p.$$

By Th. (27,2), the zeros of $f_{n-p}(z)$ all lie in the circle

$$|z| < 1 + \max [| a_k^{(n-p)} |/| a_p^{(n-p)} |], \qquad k = 0, 1, \cdots, p - 1;$$

that is, in the circle

$$|z| < 1 + M_p(r_p - 1)^{-n+p}. \tag{32,17}$$

Among these zeros is α_{n-p+1} whose modulus is r_p. This means that

$$r_p < 1 + M_p(r_p - 1)^{-n+p}; \tag{32,18}$$

i.e., that

$$(r_p - 1)^{n-p+1} < M_p,$$

$$r_p < 1 + M_p^{1/(n-p+1)}. \tag{32,19}$$

We have proved (32,19) on the assumption that $r_p > 1$. Since (32,19) is surely satisfied when $r_p \leqq 1$, we have established a result of Montel [3] and [5], as follows.

THEOREM (32,1). *At least p zeros of the polynomial* $f(z) = a_0 + a_1z + \cdots + a_nz^n$ *lie in the circle*

$$|z| < 1 + \max | a_j/a_n |^{1/(n-p+1)}, \qquad j = 0, 1, \cdots, p. \tag{32,20}$$

EXERCISES. Prove the following.

1. Th. (32,1) is a generalization of Th. (27,2).
2. If $a_0 \neq 0$, the polynomial $f(z)$ has at most p zeros in the circle

$$|z| \leqq [1 + \max (| a_{n-j} |/| a_0 |)^{1/(p+1)}]^{-1}, \qquad j = 0, 1, \cdots, n - p.$$

3. If q is an arbitrary positive integer, at least p zeros of $f(z)$ lie in the circle

$$|z| \leqq 1 + \max \left(\sum_{j=0}^{k} | a_j/a_n |^q \right)^{1/q}, \qquad k = 0, 1, \cdots, p - 1.$$

Hint: Apply the Hölder Inequality (27,10) to ineq. (32,12).

33. **Further bounds.** We shall now make some additional applications of ineq. (32,12). The first will be to the proof of a result due to Montel [3], a result similar to those in Van Vleck [3].

THEOREM (33,1). *At least p, $p \leqq n$, zeros of the polynomial* $f(z) = a_0 + a_1z + \cdots + a_nz^n$ *lie in or on the circle* $|z| = \rho$, *where* ρ *is the positive root of the equation*

$$| a_n | z^n - \sum_{k=0}^{p-1} C(n - k - 1, p - k - 1) | a_k | z^k = 0. \tag{33,1}$$

For this purpose, let us observe that

$$| f_{n-p}(z) | \geqq | a_p^{(n-p)} | | z |^p - (| a_{p-1}^{(n-p)} | | z |^{p-1} + \cdots + | a_0 |). \tag{33,2}$$

On use of (32,8) and (32,12), this inequality becomes

$$
(33,3) \qquad \begin{aligned} | f_{n-p}(z) | \geqq | a_n | | z^p | \\ - r_p^{-n+p} \sum_{k=0}^{p-1} | z |^k \sum_{j=0}^{k} C(n - p + j - 1, j) | a_{k-j} | r_p^{-j}. \end{aligned}
$$

After multiplication by r_p^{n-p} and substitution of $| z | = r_p$, the right side of (33,3) becomes

$$
F_{n-p}(r_p) = | a_n | r_p^n - \sum_{k=0}^{p-1} \sum_{j=0}^{k} C(n - p + j - 1, j) | a_{k-j} | r_p^{k-j}.
$$

A reversal of the order of summation in the sum with respect to j and a subsequent interchange of this sum with the sum with respect to k permit us to write $F_{n-p}(r_p)$ as

$$
F_{n-p}(r_p) = | a_n | r_p^n - \sum_{j=0}^{p-1} | a_j | r_p^j \sum_{k=0}^{p-j-1} C(n - p + k - 1, k).
$$

By mathematical induction, the last sum is seen to have the value $C(n - 1 - j, p - 1 - j)$. Thus,

$$
F_{n-p}(r_p) = | a_n | r_p^n - \sum_{j=0}^{p-1} C(n - 1 - j, p - 1 - j) | a_j | r_p^j .
$$

Let us now select $r_p = \rho$, the positive root of eq. (33,1). Then $F_{n-p}(\rho) = 0$. Furthermore, since eq. (33,1) has only one positive root and since $F_{n-p}(\infty) > 0$, it follows that $r_p^{n-p} | f_{n-p}(\alpha_{n-p+1}) | \geqq F_{n-p}(r_p) > 0$ for $r_p = | \alpha_{n-p+1} | > \rho$ in contradiction to the hypothesis that α_{n-p+1} is a zero of $f_{n-p}(z)$. From this result, we infer that $f(z)$ has its p zeros of smallest modulus in or on the circle $| z | = \rho$, with ρ as the positive root of eq. (33,1).

As another application of the above inequalities, let us set

$$
(33,4) \qquad N_p = \max | a_j/a_p | \qquad \text{for } j = 0, 1, \cdots , p - 1\cdot
$$

By the reasoning similar to that leading to ineq. (32,16) we may infer that for $r_p > 1$

$$
(33,5) \qquad \begin{aligned} | \alpha_1\alpha_2 \cdots \alpha_{n-p}\alpha_k^{(n-p)} | < N_p | a_p | r_p^{n-p}(r_p - 1)^{p-n}, \\ k = 0, 1, \cdots , p - 1. \end{aligned}
$$

When used in conjunction with the ineq. (33,11) developed in ex. (33,1) below, ineq. (33,5) leads to the results

$$
| \alpha_1\alpha_2 \cdots \alpha_{n-p}a_p^{(n-p)} | > | a_p | \left\{1 - N_p \sum_{1}^{\infty} C(n - p + k - 1, k)r_p^{-k}\right\},
$$

$$
(33,6) \qquad | \alpha_1\alpha_2 \cdots \alpha_{n-p}a_p^{(n-p)} | > | a_p | \{1 - N_p[r_p^{n-p}(r_p - 1)^{-n+p} - 1]\}.
$$

The division of the corresponding sides of ineqs. (33,5) and (33,6) produces the inequality

$$(33,7) \qquad | a_k^{(n-p)}/a_p^{(n-p)} | < N_p r_p^{n-p}[(1 + N_p)(r_p - 1)^{n-p} - N_p r_p^{n-p}]^{-1}.$$

We now conclude on the basis of Th. (27,2) that all the zeros of $f_{n-p}(z)$ lie in the circle

$$(33,8) \qquad | z | < 1 + \{N_p r_p^{n-p}[(1 + N_p)(r_p - 1)^{n-p} - N_p r_p^{n-p}]^{-1}\}.$$

Among these zeros is α_{n-p+1} whose modulus has been denoted by r_p. Replacing $| z |$ by r_p in (33,8), assuming the denominator on the right side of (33,8) to be positive and clearing of fractions in (33,8), we find that

$$(1 + N_p)(r_p - 1)^{n-p+1} < N_p r_p^{n-p+1},$$

and thus with $Q_p = N_p/(1 + N_p)$ and $q = 1/(n - p + 1)$ that

$$(33,9) \qquad r_p < 1/(1 - Q_p^q).$$

As may easily be verified, ineq. (33,9) is valid even if the denominator on the right side of ineq. (33,8) is zero or negative.

In summary we may state another result of Montel [3], namely

THEOREM (33,2). *At least p zeros of the polynomial* $f(z) = a_0 + a_1 z + \cdots + a_n z$ *lie in the circle*

$$| z | < 1/(1 - Q_p^q),$$

where $N_p = \max | a_j/a_p |$, $j = 0, 1, 2, \cdots, p - 1$; $Q_p = N_p/(1 + N_p)$ *and* $q = 1/(n - p + 1)$.

EXERCISES. Prove the following.

1. At least p zeros of $f(z) = \sum_0^n a_k z^k$ lie in the circle $| z | \leqq \rho$ where ρ is the positive root of the equation

$$(33,10) \qquad | a_p | \rho^p - \sum_0^{p-1} C(n - k, p - k) | a_k | \rho^k = 0.$$

Hint: From eq. (32,7), deduce the inequality

$$(33,11) \ | \alpha_1\alpha_2 \cdots \alpha_{n-p} a_p^{(n-p)} | \geqq \left\{ | a_p | - \sum_{k=1}^{p} C(n - p + k - 1, k) | a_{p-k} | r_p^{-k} \right\}$$

and substitute it into the right side of inequality (33,2) [Van Vleck 3].

2. At least p zeros of the polynomial

$$g(z) = a_0 + a_p z^p + a_{p+1} z^{p+1} + \cdots + a_n z^n, \qquad a_0 a_p a_n \neq 0,$$

lie (a) in the circle $| z | \leqq [C(n - 1, p - 1) | a_0/a_n |]^{1/n}$; (b) in the circle $| z | \leqq [C(n, p) | a_0/a_p |]^{1/p}$. Both limits are attainable [Van Vleck 3].

3. Th. (33,2) reduces to Th. (27,2) when $p = n$.

4. At least p zeros of $f(z) = \sum_0^n a_k z^k$ lie in the circle $|z| \leqq 2(n - p + 1)A_p$, where $A_p = \max |a_k/a_{k+1}|$ for $k = 0, 1, 2, \cdots, p - 1$. Hint: Apply Th. (33,2) to the polynomial $P(\zeta) = f(A_p\zeta)$, noting that, since

$$1/2 \leqq [1 - (1/2q)]^q \qquad \text{for } q = 1, 2, \cdots,$$

we may write

$$(1 - 2^{-1/q})^{-1} \leqq 2q \qquad \text{[Montel 3].}$$

5. At least one zero of the polynomial $f(z) = a_0 + a_1 z + \cdots + a_n z^n$, $a_0 \neq 0$, lies in each of the four circles $|z| \leqq r_k$ with

$$r_1 = |na_0/a_1|, \qquad r_2 = |na_0/(2a_0a_2 - a_1^2)|^{1/2},$$

$$r_3 = |na_0^3/(3a_0^2a_3 - 3a_0a_1a_2 + a_1^3)|^{1/3}, \qquad r_4 = |na_0^4/(4a_0^3a_4 - 4a_0^2a_1a_3 - 2a_0^2a_2^2 - 2a_0a_1^2a_2 - a_1^4)|^{1/4}.$$

Hint: Use ex. (14,2); evaluate right side of eq. (14,7) and thus r_p for $z_0 = 0$ and $p = 1, 2, 3, 4$ [Nagy 6 and 12].

6. At least one zero of $f(z) = a_0 + a_1 z + a_{k+1}z^{k+1} + a_{k+2}z^{k+2} + \cdots + a_n z^n$, $a_0 a_1 \neq 0$, lies in the circle $|z| \leqq n^{1/k} |a_1/a_0|$ [Nagy 12].

7. At least one zero of $f(z) = a_0 + a_p z^p + a_{p+1}z^{p+1} + \cdots + a_n z^n$, $1 \leqq p$, $a_0 a_{p+h} \neq 0$, lies in the circle $|z| \leqq |na_0/(p + h)a_{p+h}|^{1/(p+h)}$, $h = 0, 1, 2, \cdots, p - 1$ [Carmichael-Mason 1, when $h = 0$; Nagy 12, when $0 \leqq h < p$].

34. **Lacunary polynomials.** In sections 32 and 33, we found that, when the coefficients $a_0, a_1, \cdots, a_p$ are fixed but the remaining a_j, $j > p$, are arbitrary, there exist various circles $|z| \leqq r$ which contain at least p zeros of the polynomial. If in addition we were to fix some of the coefficients a_j, $j > p$, we should obviously find that the resulting polynomials have p zeros in circles $|z| \leqq r_1$, with $r_1 \leqq r$.

An important class of such polynomials are those of the lacunary type

$$(34,1) \qquad \begin{gathered} f(z) = a_0 + a_1 z + \cdots + a_p z^p + a_{n_1}z^{n_1} + a_{n_2}z^{n_2} + \cdots + a_{n_k}z^{n_k}, \\ 0 < n_0 = p < n_1 < n_2 < \cdots < n_k, \qquad a_0 a_p \neq 0. \end{gathered}$$

Here, compared with eq. (31,1), the coefficients a_j, $0 \leqq j \leqq p$, are fixed; the coefficients a_{n_j}, $j = 1, 2, \cdots, k$, are arbitrary and the remaining coefficients a_j are zero.

As we stated in sec. 31, Landau [1] and [2] initiated the study of polynomials of this form in 1906-7. He considered the cases $p = 1$, $k = 1$ or 2, proving for these cases the existence of a circle $|z| = R(a_0, a_1)$ containing at least one zero of $f(z)$. He also raised the question as to whether or not a circle with this same property existed in the case $p = 1$ and k arbitrary.

An affirmative reply was given in 1907 by Allardice [1] who proved that, when $p = 1$, at least one zero of $f(z)$ lies in the circle

$$|z| \leqq |a_0/a_1| \prod_{j=1}^{k} [n_j/(n_j - 1)]$$

and by Fejér [1] who proved that, when $a_1 = a_2 = \cdots = a_{p-1} = 0$, at least one zero of $f(z)$ lies in the circle

$$|z| \leqq \left\{ |a_0/a_p| \prod_{j=1}^{k} [n_j/(n_j - p)] \right\}^{1/p}. \tag{34,2}$$

About sixteen years later, Montel [1] proved that for any polynomial (34,1) there exists a circle $|z| \leqq R(a_0, a_1, \cdots, a_p, k)$ containing at least p zeros of $f(z)$ and Walsh [10] proved that, when $a_1 = a_2 = \cdots = a_{p-1} = 0$ and $a_u \neq 0$ for some $u = n_h$, $0 \leqq h \leqq k$, there exists a circle $|z| \leqq R(a_0, a_u, k)$ containing at least p zeros of $f(z)$. As to the specific determination of the radii of these circles, Montel [1] showed that, when $p = 2$ and $a_1 = 0$, at least two zeros of $f(z)$ lie in the circle

$$|z| \leqq [|a_0/a_2| k(k+1)/2]^{1/2} = b,$$

the limit being attained for $a_0 = a_2 = 1$ by each of the two polynomials $[(1 \pm iz/b)^k(1 \mp ikz/b)]$. In 1925 Van Vleck [3] established that, when $a_1 = a_2 = \cdots = a_{p-1} = 0$, $n_j = p + j$, $j = 1, 2, \cdots, k$, and $a_{p+h} \neq 0$ for some h, $0 \leqq h \leqq k$, at least p zeros of $f(z)$ lie in the circle

$$|z| \leqq [C(p + h - 1, p - 1)C(n, p + h) \, |a_0/a_{p+h}|]^{1/(p+h)},$$

the limit being attained for $h = 0$ by the polynomial

$$(z - b)^{n-p+1} \sum_{0}^{p-1} C(n - p + j, j)(z/b)^j$$

where b is a pth root of $[(-1)^{p-1}C(n, p)a_0/a_p]$. In 1928 Biernacki [1] proved that, when $a_1 = a_2 = \cdots = a_{p-1} = 0$, at least p zeros of $f(z)$ lie in the circle (34,2), the limit being attained for $n_j = p + j$, $j = 1, 2, \cdots, k$, and that, if all the zeros of the polynomial

$$f_0(z) = a_0 + a_1 z + \cdots + a_p z^p$$

lie in the circle $|z| \leqq R_0$, at least p zeros of $f(z)$, eq. (34,1), lie in the circle

$$|z| \leqq R_0 \prod_{j=1}^{k} [n_j/(n_j - p)], \tag{34,3}$$

the limit being attained only for $p = k = 1$.

These results are in agreement with that of Dieudonné [7] which for a_0,

$a_1, \cdots, a_{p-1}$ and a_u, $u = n_h$, fixed states that the smallest circle $|z| \leqq r$ containing $q \leqq p$ zeros of $f(z)$ has a radius of the order of magnitude $r_q(n) = O(n^{1/(p-1+q)})$ in general.

The more general of the above limits, however, require complicated derivations. For this reason we shall devote the next two sections to the construction of alternative limits which, though less exact, are much simpler to establish.

Our first theorem in this direction will be obtained by the use of some previous results on the zeros of the derivative of a polynomial, specifically ex. (6.4), Th. (25,4), Th. (26,2) and ex. (25,2). These results state in effect, first, that, if z_1 is a zero of $f'(z)$, at least one zero of $f(z)$ lies in the region $|z| \geqq |z_1|$, and, secondly, that, if at most $p - 1$ zeros of $f'(z)$ lie in a circle $|z| \leqq r$, then at most p zeros of $f(z)$ lie in the circle

$$|z| \leqq r/\phi(n, p + 1). \tag{34,4}$$

Among the known functions $\phi(n, p)$ are those given in Ths. (25,4) and (26,2); namely,

$$\phi(n, p) = \csc \pi/2(n - p + 1) \tag{34,5}$$

and

$$\phi(n, p) = \prod_{j=1}^{n-p} (n + j)/(n - j). \tag{34,6}$$

We shall apply these theorems to the polynomial

$$F(z) = z^{n_k} f(1/z) = \sum_{i=0}^{p} a_i z^{n_k - i} + \sum_{i=1}^{k} a_{n_i} z^{n_k - n_i} \tag{34,7}$$

and to the other polynomials of the sequence $F_j(z)$ defined by the equations

$$F_0(z) = F(z), \tag{34,8}$$

$$F_j'(z) = z^{n_{k-j} - n_{k-j-1} - 1} F_{j+1}(z), \qquad j = 0, 1, \cdots, k - 1. \tag{34,9}$$

By straightforward computation, we may show that

$$\begin{aligned} F_j(z) = {} & \sum_{i=0}^{p} (n_k - i)(n_{k-1} - i) \cdots (n_{k-j+1} - i) a_i z^{n_{k-j} - i} \\ & + \sum_{i=1}^{k-j} (n_k - n_i)(n_{k-1} - n_i) \cdots (n_{k-j+1} - n_i) a_{n_i} z^{n_{k-j} - n_i}. \end{aligned} \tag{34,10}$$

In particular, we find that

$$F_k(z) = \sum_{i=0}^{p} (n_k - i)(n_{k-1} - i) \cdots (n_1 - i) a_i z^{p-i}. \tag{34,11}$$

Let us also define

$$(34,12) \qquad f_k(z) = z^p F_k(1/z) = \sum_{i=0}^{p} (n_k - i)(n_{k-1} - i) \cdots (n_1 - i)a_i z^i.$$

Since $a_0 \neq 0$, $f_k(z)$ does not vanish at the origin. Let us denote by ρ_1 the largest and ρ_2 the smallest positive number such that all the zeros of $f_k(z)$ lie in the annular ring $0 < \rho_1 \leqq |z| \leqq \rho_2$. Being according to (34,12) the reciprocals of the zeros of $f_k(z)$, the zeros of $F_k(z)$ lie in the ring $1/\rho_2 \leqq |z| \leqq 1/\rho_1$, with at least one zero of $F_k(z)$ on each of the circles $|z| = 1/\rho_2$ and $|z| = 1/\rho_1$. According to (34,9) the zeros of $F'_{k-1}(z)$ are those of $F_k(z)$ and a zero of multiplicity $n_1 - p - 1$ at $z = 0$; thus $F'_{k-1}(z)$ has at least one zero on $|z| = 1/\rho_1$ and exactly $n_1 - p - 1$ in $|z| < 1/\rho_2$. Consequently, $F_{k-1}(z)$ has at least one zero in $|z| \geqq 1/\rho_1$ and at most $n_1 - p$ zeros in

$$(34,13) \qquad |z| < [\rho_2\phi(n_1, n_1 - p + 1)]^{-1}.$$

Similarly, since the zeros of $F'_{k-2}(z)$ are the zeros of $F_{k-1}(z)$ and a zero of multiplicity $n_2 - n_1 - 1$ at $z = 0$, $F'_{k-2}(z)$ has at least one zero in $|z| \geqq 1/\rho_1$ and at most $n_2 - p - 1$ zeros satisfying (34,13). Consequently, $F_{k-2}(z)$ has at least one zero in $|z| \geqq 1/\rho_1$ and at most $n_2 - p$ zeros in

$$(34,14) \qquad |z| < [\rho_2\phi(n_1, n_1 - p + 1)\phi(n_2, n_2 - p + 1)]^{-1}.$$

Continuing in this manner, we may by induction demonstrate that $F(z)$ has at least one zero in $|z| \geqq 1/\rho_1$ and at most $n_k - p$ zeros in

$$(34,15) \qquad |z| < [\rho_2\phi(n_1, n_1 - p + 1)\phi(n_2, n_2 - p + 1) \cdots \phi(n_k, n_k - p + 1)]^{-1}.$$

Finally, in view of eq. (34,7) by which the zeros of $f(z)$ are defined as the reciprocals of the zeros of $F(z)$, we conclude that $f(z)$ has at least one zero in $|z| \leqq \rho_1$ and at most $n_k - p$ zeros in

$$|z| > \rho_2\phi(n_1, n_1 - p + 1)\phi(n_2, n_2 - p + 1) \cdots \phi(n_k, n_k - p + 1).$$

Hence, $f(z)$ has at least p zeros in

$$(34,16) \qquad |z| \leqq \rho_2\phi(n_1, n_1 - p + 1)\phi(n_2, n_2 - p + 1) \cdots \phi(n_k, n_k - p + 1).$$

These results which are due to Marden [14] may be summarized in the form of

THEOREM (34,1). *Given the polynomial*

$$(34,17) \qquad f(z) = a_0 + a_1 z + \cdots + a_p z^p + a_{n_1} z^{n_1} + a_{n_2} z^{n_2} + \cdots + a_{n_k} z^{n_k}$$

with $0 < n_0 = p < n_1 < n_2 < \cdots < n_k$ *and* $a_0 a_p \neq 0$. *Let*

$$(34,18) \qquad f_k(z) = n_1 n_2 \cdots n_k a_0 + (n_1 - 1)(n_2 - 1) \cdots (n_k - 1)a_1 z + \cdots + (n_1 - p)(n_2 - p) \cdots (n_k - p)a_p z^p.$$

If ρ_1 *is the largest and* ρ_2 *the smallest positive number such that all the zeros of* $f_k(z)$

lie in $\rho_1 \leqq |z| \leqq \rho_2$, *then* $f(z)$ *has at least one zero in the circle* $|z| \leqq \rho_1$ *and at least p zeros in the circle*

$$|z| \leqq \rho_2\phi(n_1, n_1 - p + 1)\phi(n_2, n_2 - p + 1) \cdots \phi(n_k, n_k - p + 1). \tag{34,19}$$

Using the known $\phi(n, p)$ as given in (34,5) and (34,6) we deduce the following limits due to Marden [14].

COROLLARY (34,1a). *In the notation of Th.* (34,1), *at least p zeros of* $f(z)$ *lie in each of the circles*

$$|z| \leqq \rho_2 \csc^k (\pi/2p), \tag{34,20}$$

$$|z| \leqq \rho_2 \prod_{i=1}^{k} \prod_{j=1}^{p-1} (n_i + j)/(n_i - j). \tag{34,21}$$

In particular if $a_1 = a_2 = \cdots = a_{p-1} = 0$, the zeros of $f_k(z)$ all have the modulus

$$\left[\frac{n_1 n_2 \cdots n_k}{(n_1 - p)(n_2 - p) \cdots (n_k - p)} \left|\frac{a_0}{a_p}\right|\right]^{1/p} = \rho_1 = \rho_2 .$$

Thus, from Th. (34,1) and Cor. (34,1a) we infer

COROLLARY (34,1b). *At least one zero of the polynomial*

$$f(z) = a_0 + a_p z^p + a_{n_1} z^{n_1} + \cdots + a_{n_k} z^{n_k}, \tag{34,22}$$

$$0 < p < n_1 < n_2 < \cdots < n_k, \qquad a_0 a_p \neq 0,$$

lies in the circle

$$|z| \leqq \left[\frac{n_1 n_2 \cdots n_k}{(n_1 - p)(n_2 - p) \cdots (n_k - p)} \left|\frac{a_0}{a_p}\right|\right]^{1/p} = R \tag{34,23}$$

and at least p zeros lie in each of the circles

$$|z| \leqq R \csc^k (\pi/2p), \tag{34,24}$$

$$|z| \leqq R \prod_{j=1}^{k} \prod_{i=1}^{p-1} (n_i + j)/(n_i - j). \tag{34,25}$$

Limit (34,23) is due to Féjer [1] and (34,24) and (34,25) are due to Marden [14].

The inequalities (34,23) to (34,25) may be replaced by inequalities which are simpler though not as sharp. We note that

$$N(p, k) = \frac{n_1 n_2 \cdots n_k}{(n_1 - p)(n_2 - p) \cdots (n_k - p)}$$

$$= \left[\left(1 - \frac{p}{n_1}\right)\left(1 - \frac{p}{n_2}\right) \cdots \left(1 - \frac{p}{n_k}\right)\right]^{-1}$$

and that accordingly the fraction may be maximized by giving the n_k their minimum values $n_k = p + k$; viz.,

$$N(p, k) \leqq C(p + k, k). \tag{34,26}$$

Furthermore, if $p < k$, the right side of (34,26) may be written as

$$C(p + k, p) \leqq [(k + 1)(2k + 2) \cdots (pk + p)/1 \cdot 2 \cdots p] = (k + 1)^p,$$

the equality sign holding only when $p = 1$. The right side of (34,26) may be treated similarly when $p \geqq k$. Thus, in all cases,

$$N(p, k) \leqq (k + 1)^p, \tag{34,27}$$

the equality sign holding only when $p = 1$.

Taking (34,26) and (34,27) into consideration, we may restate Cor. (34,1b), following Marden [14], as

COROLLARY (34,1c). *The polynomial*

$$f(z) = a_0 + a_p z^p + a_{n_1} z^{n_1} + \cdots + a_{n_k} z^{n_k}$$

$$p < n_1 < n_2 < \cdots < n_k, \qquad a_0 a_p \neq 0,$$

has at least one zero in the circle

$$|z| \leqq [C(p + k, k) \, |a_0/a_p|]^{1/p} = R_1 \leqq (k + 1) \, |a_0/a_p|^{1/p} = R_2$$

and at least p zeros in the circles

$$|z| \leqq R_1 \csc^k (\pi/2p) \leqq R_2 \csc^k (\pi/2p),$$

$$|z| \leqq R_1 \prod_{i=1}^{k} \prod_{j=1}^{p-1} (n_i + j)/(n_i - j) \leqq R_2 \prod_{i=1}^{k} \prod_{j=1}^{p-1} (n_i + j)/(n_i - j).$$

EXERCISES. Prove the following.

1. The polynomial (34,17) has at least one zero in the circle $|z| \leqq \rho_1$, where ρ_1 is the positive root of the equation

$$- n_1 n_2 \cdots n_k \, |a_0| + (n_1 - 1)(n_2 - 1) \cdots (n_k - 1) \, |a_1| \, z + \cdots$$
$$+ (n_1 - p)(n_2 - p) \cdots (n_k - p) \, |a_p| \, z^p = 0,$$

and at least p zeros in the circle

$$|z| \leqq \rho_2 \phi(n_1, n_1 - p + 1) \phi(n_2, n_2 - p + 1) \cdots \phi(n_k, n_k - p + 1),$$

where ρ_2 is the positive root of the equation

$$n_1 n_2 \cdots n_k \, |a_0| + (n_1 - 1)(n_2 - 1) \cdots (n_p - 1) \, |a_1| \, z + \cdots$$
$$+ (n_1 - p + 1)(n_2 - p + 1) \cdots (n_k - p + 1) \, |a_{p-1}| \, z^{p-1}$$
$$- (n_1 - p)(n_2 - p) \cdots (n_k - p) \, |a_p| \, z^p = 0.$$

Hint: Use Ths. (27,1) and (34,1); cf. ex. (33,1) [Marden 14].

2. The polynomial (34,17) has at least p zeros in the circle

$$|z| \leqq \rho_2 \phi(n_1, n_1 - p + 1) \cdots \phi(n_k, n_k - p + 1)$$

where

$$\rho_2 = 1 + \max \left[|a_j/a_p| \prod_{i=1}^{k} (n_i - j)/(n_i - p) \right], \tag{34,28}$$

$$j = 0, 1, \cdots, p - 1.$$

Hint: Use Th. (27,2) [Marden 14].

3. The ρ_2 in eq. (34,28) satisfies the inequality

$$\rho_2 \leqq 1 + M_p C(p + k, k) \leqq 1 + (k + 1)^p M_p$$

where $M_p = \max |a_j/a_p|, j = 0, 1, \cdots, p - 1$.

4. At least one zero of polynomial

$$f(z) = a_0 + a_1 z^{n_1} + a_2 z^{n_2} + \cdots + a_k z^{n_k} \tag{34,29}$$

lies in the circle $|z| \leqq \rho$ where

$$\rho = \min \left[|a_0/a_{n_j}| \prod_{i=1}^{k-j} n_{j+i}/(n_{j+i} - n_j) \right]^{1/n_j},$$

$$j = 1, 2, \cdots, k - 1 \qquad \text{[Fekete 4].}$$

5. All the zeros of eq. (34,29) lie in the circle

$$|z| \leqq r, r = \max [|a_0/a_1|^{1/n_1}, |2a_{n_j}/a_{n_{j+1}}|^{1/(n_{j+1}-n_j)}], \quad j = 1, 2, \cdots, k - 1.$$

Hint: Use the method of ex. (30,6).

6. If in ex. 5 the coefficients a_j satisfy a linear relation $\lambda_0 a_0 + \lambda_1 a_1 + \cdots + \lambda_k a_k = 0$, with $\lambda_0 \neq 0$, then $f(z)$ has at least one zero in the circle $|z| \leqq 2kM$, where $M = \max (1, |\lambda_j/\lambda_0|), j = 1, 2, \cdots, k$. Hint: Show $|a_0/a_j| \leqq 2^{n_j} M$ for at least one j, since otherwise $|a_0| > |a_0| (2^{-n_1} + 2^{-n_2} + \cdots + 2^{-n_k}) > M(|a_1| + |a_2| + \cdots + |a_k|)$ in contradiction with the relation $\sum \lambda_j a_j = 0$ [Fekete 4].

7. At least one zero of the polynomial (34,17) lies in each circle which has as diameter the line-segment joining point $z = 0$ with any of p zeros of (34,18). [Fejér 1a].

8. The polynomial

$$f(z) = a_0 \sum_{j=0}^{p} [z^j/(n_1 - j)(n_2 - j) \cdots (n_k - j)] + \sum_{j=1}^{k} a_j z^{n_j}$$

has at least p zeros in the circle $|z| \leqq \csc^k (\pi/2p)$.

35. Other bounds for lacunary polynomials. A theorem similar to Th. (34,1) but in which the polynomial $f_k(z)$ in eq. (34,18) is replaced by one involving the first $p + 1$ terms of $f(z)$ will now be established with the aid of ex. (16,11).

In ex. (16,11) let us choose $m = p$, $n = k$, $f(z) = P(z) = a_0 + a_1z + \cdots + a_pz^p$, and $g(z) = (n_1 - z)(n_2 - z) \cdots (n_k - z)$. Then $h(z) = f_k(z)$ and

$$| g(0)/g(m) | = \prod_{j=1}^{k} n_j/(n_j - p) > 1.$$

If R is the radius of the smallest circle $| z | = R$ which contains all the zeros of $P(z)$, we learn from ex. (16,11) on setting $r_1 = 0$ and $r_2 = R$ that all the zeros of $f_k(z)$ lie in the circle

$$| z | \leqq R \prod_{j=1}^{k} n_j/(n_j - p) = R'.$$

In view of the definition of ρ_2 as the radius of the smallest circle $| z | \leqq r$ containing all the zeros of $f_k(z)$, we conclude that $\rho_2 \leqq R'$.

In place of Th. (34,1), we may therefore state the following theorem, in some respects simpler, but not sharper than Th. (34,1).

THEOREM (35,1). *If all the zeros of the polynomial*

$$P(z) = a_0 + a_1z + \cdots + a_pz^p \tag{35,1}$$

lie in the circle $| z | \leqq R$, *at least* p *zeros of the polynomial*

$$f(z) = a_0 + a_1z + \cdots + a_pz^p + a_{n_1}z^{n_1} + \cdots + a_{n_k}z^{n_k},$$

$$0 < p < n_1 < \cdots < n_k, \qquad a_0a_p \neq 0,$$

lie in the circle

$$| z | \leqq R \prod_{j=1}^{k} n_j\phi(n_j, n_j - p + 1)/(n_j - p). \tag{35,2}$$

Using the values of $\phi(n, p)$ in eqs. (34,5) and (34,6), we may replace ineq. (35,2) by the more specific ones

$$| z | \leqq R\left[\prod_{j=1}^{k} n_j/(n_j - p)\right] \csc^k \pi/2p; \tag{35,3}$$

$$| z | \leqq R \prod_{j=1}^{k} \frac{n_j}{n_j - p} \prod_{i=1}^{p-1} \frac{n_j + i}{n_j - i} = R \prod_{j=1}^{k} \prod_{i=0}^{p-1} \frac{n_j + i}{n_j + i - p}, \tag{35,4}$$

the latter being due to Biernacki [3]. The right sides of (35,2) and (35,3) both have values in excess of R. That they may not be replaced by values less than R is clear from the fact that $P(z)$ is one of the polynomials $f(z)$; that is, the $f(z)$ with $a_j \to 0$, all $j > p$.

It is known, however, that the right side of ineq. (35,3) may be replaced by the smaller bound (34,3), a bound whose derivation is quite complicated. But neither this bound nor those given in (35,3) or (35,4) is known to be attained

by at least one of the p zeros of smallest modulus for at least one polynomial $f(z)$ of type (34,17). In other words, none of the bounds is as yet known to be the best possible one.

Of the two bounds (35,3) and (35,4), the second has the advantage that, as $k \to \infty$, it approaches a finite limit, provided the series $\sum 1/n_i$ converges. This fact suggests the following theorem of the Picard type, due to Biernacki [1] and [3].

THEOREM (35,2). *If the series $\sum 1/m_i$ converges, the entire function*

$$f(z) = a_0 + a_{m_1}z^{m_1} + a_{m_2}z^{m_2} + \cdots, \qquad 0 < m_1 < m_2 < \cdots,$$

if not identically zero, takes on every finite value A an infinite number of times.

To prove this theorem, let us choose p as any of the numbers $m_1, m_2, \cdots,$ and form the polynomial

$$Q_k(z) = (a_0 - A) + a_{m_1}z^{m_1} + \cdots + a_pz^p + a_{n_1}z^{n_1} + \cdots + a_{n_k}z^{n_k}$$

in which the last k terms are the k terms following a_pz^p in $f(z)$. Let us denote by R the radius of the circle $|z| \leqq R$ in which lie the zeros of the polynomial

$$Q_0(z) = (a_0 - A) + a_{m_1}z^{m_1} + \cdots + a_pz^p.$$

By Th. (35,1), $Q_k(z)$ has at least p zeros in the circle (35,4). The right side of (35,4) may be written as

$$R \prod_{j=1}^{k} \prod_{i=0}^{p-1} [1 - (p/(n_j + i))]^{-1} < R \prod_{i=0}^{p-1} \prod_{j=1}^{\infty} [1 - (p/(n_j + i))]^{-1} = R_1 .$$

The infinite products occurring in R_1 converge due to the convergence of the series $\sum 1/m_i$. That is, R_1 is a number independent of k such that in the circle $|z| = R_1$ lie at least p zeros of $Q_k(z)$.

On the other hand, the terms in $Q_k(z)$ are the terms of $f(z) - A$ up to that in z^{n_k}. Since $f(z)$ is an entire function, $Q_k(z)$ converges uniformly to $f(z) - A$ in any circle $|z| \leqq R_1 + \epsilon$, $\epsilon > 0$. But, by Hurwitz' Theorem (Th. 1,5), given any sufficiently small positive ϵ, there is at least one zero of $f(z) - A$ in each of the p circles of radius ϵ drawn about the p zeros of $Q_k(z)$ in $|z| \leqq R_1$. Hence, there are at least p zeros of $f(z) - A$ in the circle $|z| \leqq R_1 + \epsilon$. Due to the fact that p is an arbitrary m_i, we conclude that $f(z) - A$ has an infinite number of zeros. That is, $f(z)$ assumes the value A an infinite number of times.

EXERCISES. Prove the following.

1. At least p zeros of each of the two polynomials

$$1 + z^p + a_1z^{n_1} + \cdots + a_kz^{n_k},$$

$$1 + z + z^2 + \cdots + z^p + a_1z^{n_1} + \cdots + a_kz^{n_k}$$

lie in each of the circles (35,3) and (35,4) with $R = 1$.

2. The polynomial

$$P(z) = 1 + z^p + a_1 z^{p+q} + a_2 z^{p+2q} + \cdots + a_k z^{p+kq},$$

where $p \geqq p_0$ and q is not a factor of p, has at least p_0 zeros in a circle $|z| \leqq R(p_0)$ [Landau 2, case $p_0 = 1$; Montel 1, p_0 arbitrary].

3. The trinomial $1 + x^p + ax^n$ has at least one zero in each of the sectors

$$|(\arg z) - (2k+1)(\pi/p)| \leqq \pi/n, \qquad k = 0, 1, \cdots, p-1.$$

The limits are attained when $a = p(n-p)^{(n-p)/p}/\omega^n n^{n/p}$ where ω is any pth root of (-1) [Nekrasoff 1, Kempner 5, Herglotz 1, Biernacki 1].

4. If $n_2 \geqq 3n_1/2$, the quadrinomial

$$1 + x^p + a_1 x^{n_1} + a_2 x^{n_2}, \qquad p < n_1 < n_2,$$

has at least one zero in each of the sectors

$$|(\arg z) - (2k+1)/(\pi/p)| \leqq (\pi/n_1), \qquad k = 0, 1, \cdots, p-1.$$

The limits are attained when for $k = 1, 2$

$$a_k = -\{(-1)^k p[(n_1 - p)(n_2 - p)]^{n_k/p}\}/\{(n_2 - n_1)(n_k - p)(n_1 n_2)^{(n_k - p)/p} n_k \omega^{n_k}\},$$

where ω is a pth root of (-1) [Biernacki 1, pp. 603-613].

5. Every quadrinomial $1 + az^p + z^{2p} + bz^n$, $n > 2p$, a and b arbitrary, has at least $p - 1$ zeros in the circle $|z| \leqq 1$ [Dieudonné 6].

CHAPTER IX

THE NUMBER OF ZEROS IN A HALF-PLANE OR SECTOR

36. **Dynamic stability.** The problem discussed in the last two chapters, the determination of bounds for some or all of the zeros of a polynomial $f(z)$, may be regarded as the problem of finding a region (a circle $|z| = R$) which will contain a prescribed number p of zeros of $f(z)$. The converse type of problem is of equal importance. It is the problem of finding the exact or approximate number of zeros which lie in a prescribed region such as a half-plane, a sector or a circular region.

In order to see how this sort of problem arises in applied mathematics, let us as in Routh [1] and [2] consider the example of a particle of unit mass moving in the x, y plane subject to a resultant force with the x-component $X(x, y, t, u, v)$ and y-component $Y(x, y, t, u, v)$, where (x, y) and (u, v) denote respectively the coordinates and the velocity components of the particle. Let us assume that X and Y possess continuous first partial derivatives in the neighborhood of some value $(x_0, y_0, 0, u_0, v_0)$. The equations of motion are then

$$du/dt = X(x, y, t, u, v), \qquad dv/dt = Y(x, y, t, u, v). \tag{36,1}$$

Let us denote by $x = x_1(t)$ and $y = y_1(t)$ the solutions corresponding to the set of initial conditions $x(0) = x_0$, $y(0) = y_0$, $u(0) = u_0$, $v(0) = v_0$, and by $x = x_1(t) + \xi(t)$ and $y = y_1(t) + \eta(t)$ the solutions corresponding to the slightly altered (i.e. disturbed) set of initial conditions $x(0) = x_0 + \xi_0$, $y(0) = y_0 + \eta_0$, $u(0) = u_0 + \sigma_0$, $v(0) = v_0 + \tau_0$.

Substituting these solutions into eq. (36,1), subtracting the resulting equations and setting $\sigma = d\xi/dt$ and $\tau = d\eta/dt$, we find for ξ and η the differential equations

$$\begin{aligned} d\sigma/dt &= X(x_1 + \xi, y_1 + \eta, t, u_1 + \sigma, v_1 + \tau) - X(x_1, y_1, t, u_1, v_1) \\ &= X_x\xi + X_y\eta + X_u\sigma + X_v\tau, \\ d\tau/dt &= Y(x_1 + \xi, y_1 + \eta, t, u_1 + \sigma, v_1 + \tau) - Y(x_1, y_1, t, u_1, v_1) \\ &= Y_x\xi + Y_y\eta + Y_u\sigma + Y_v\tau, \end{aligned}$$

where the partial derivatives of X and Y are formed for the intermediate value $(x_1 + \theta\xi, y_1 + \theta\eta, t, u_1 + \theta\sigma, v_1 + \theta\tau)$ with $0 \leqq \theta \leqq 1$. If ξ_0, η_0, σ_0 and τ_0 are all sufficiently small, we may with good approximation compute ξ and η by means of the equations

$$\text{(36,2)}\qquad d\sigma/dt = A_1\xi + A_2\eta + A_3\sigma + A_4\tau, \qquad d\tau/dt = B_1\xi + B_2\eta + B_3\sigma + B_4\tau,$$

where the coefficients are the real constants obtained on evaluating the above partial derivatives for $(x_0, y_0, 0, u_0, v_0)$. As is well known, eqs. (36,2) have solutions of the form

$$\xi = \lambda e^{\gamma t}, \qquad \eta = \mu e^{\gamma t}.$$

On substituting these into eqs. (36,2), we obtain for γ, λ and μ the equations

$$\lambda(\gamma^2 - A_3\gamma - A_1) = \mu(A_4\gamma + A_3), \qquad \lambda(B_3\gamma + B_1) = \mu(\gamma^2 - B_4\gamma - B_3),$$

and, by eliminating μ and λ, we obtain for γ a fourth degree equation with real coefficients

$$f(\gamma) = a_0 + a_1\gamma + a_2\gamma^2 + a_3\gamma^3 + a_4\gamma^4 = 0.$$

In general, the roots of this equation will be distinct complex numbers $\gamma_k = \alpha_k \pm i\beta_k$, $k = 1, 2$, so that the general solution of the system (36,2) will be

$$\text{(36,3)}\qquad \begin{aligned} \xi &= e^{\alpha_1 t}(\lambda_1 \sin \beta_1 t + \lambda_2 \cos \beta_1 t) + e^{\alpha_2 t}(\lambda_3 \sin \beta_2 t + \lambda_4 \cos \beta_2 t),\\ \eta &= e^{\alpha_1 t}(\mu_1 \sin \beta_1 t + \mu_2 \cos \beta_1 t) + e^{\alpha_2 t}(\mu_3 \sin \beta_2 t + \mu_4 \cos \beta_2 t). \end{aligned}$$

The original solution $x = x_1(t)$, $y = y_1(t)$ is said to be stable if the disturbance functions $\xi(t)$, $\eta(t)$ approach zero as $t \to \infty$. According to eqs. (36,3), this occurs if $\alpha_1 < 0$ and $\alpha_2 < 0$. For stability it is therefore sufficient that all four roots of $f(\gamma) = 0$ have negative real parts. That is, it is sufficient that all four zeros of the polynomial $f(\gamma)$ lie in the left half-plane.

For a more detailed review of the problem of stability in relation to the distribution of the zeros of a polynomial, the reader is referred to Bateman [1].

EXERCISE. Prove the following.

1. In sec. 36 let $z = x + iy$, $\zeta = \xi + i\eta$, $w = dz/dt$, $\rho = d\zeta/dt$ and

$$Z(z, t, w) = X(x, y, t, u, v) + iY(x, y, t, u, v).$$

If Z is an analytic function of z and w in the neighborhood of $(z_0, 0, w_0)$, then eqs. (36,2) may be replaced by an equation of the form

$$\text{(36,4)}\qquad d\rho/dt = M\rho + N\zeta$$

where M and N are complex constants. The particle will have a stable motion if both zeros of the polynomial $f(z) = \gamma^2 - M\gamma - N$ lie in the left half-plane.

37. **Cauchy indices.** We shall now proceed to the determination of the number of zeros of a polynomial in a given half-plane. For simplicity we shall at first assume the half-plane to be the upper half-plane $\mathfrak{I}(z) > 0$.

Our method will consist in applying Th. (1,6) to the case that line L is the axis of reals and the direction for traversing L is from $z = -\infty$ to $z = +\infty$. Hence, if $f(z)$ has no zeros on the real axis, the numbers p and q,

$$p = (1/2)[n + (1/\pi)\Delta_L \arg f(z)], \qquad q = (1/2)[n - (1/\pi)\Delta_L \arg f(z)],$$

are the number of zeros which $f(z)$ has in the upper and lower half-planes respectively.

We shall find it convenient to throw $f(z)$ into the form

$$f(z) = a_0 + a_1 z + \cdots + a_{n-1}z^{n-1} + z^n \tag{37,1}$$

and write

$$a_k = a_k' + ia_k'', \qquad k = 0, 1, \cdots, n-1, \tag{37,2}$$

where a_k' and a_k'' are real and not all a_k'' are zero. Then on the x-axis

$$f(x) = P_0(x) + iP_1(x), \tag{37,3}$$

where

$$\begin{aligned} P_0(x) &= a_0' + a_1'x + \cdots + a_{n-1}'x^{n-1} + x^n, \\ P_1(x) &= a_0'' + a_1''x + \cdots + a_{n-1}''x^{n-1}. \end{aligned} \tag{37,4}$$

Furthermore, on the x-axis

$$\arg f(x) = \operatorname{arc\,cot} \rho(x), \qquad \rho(x) = P_0(x)/P_1(x). \tag{37,5}$$

In order to calculate the net change in $\arg f(x)$, let us denote the real distinct zeros of $P_0(x)$ by $x_1, x_2, \cdots, x_\nu$ ($\nu \leqq n$) and let us assume that these are arranged in increasing order,

$$x_1 < x_2 < \cdots < x_\nu . \tag{37,6}$$

Since $f(x) \neq 0$, no x_k is also a zero of $P_1(x)$. From the graph of arc cot ρ, we infer that the change $\Delta_k \arg f(x)$ in $\arg f(z)$ as $z = x$ varies from $x_k + \epsilon$ to $x_{k+1} - \epsilon$, ϵ being a sufficiently small positive number, will according to eq. (37,5) have the values

$$\begin{aligned} &\Delta_k \arg f(z) = -\pi \quad &&\text{if} \quad \rho(x_k + \epsilon) > 0 \quad \text{and} \quad \rho(x_{k+1} - \epsilon) < 0, \\ &\Delta_k \arg f(z) = +\pi \quad &&\text{if} \quad \rho(x_k + \epsilon) < 0 \quad \text{and} \quad \rho(x_{k+1} - \epsilon) > 0, \\ &\Delta_k \arg f(z) = 0 \quad &&\text{if} \quad \rho(x_k + \epsilon)\rho(x_{k+1} - \epsilon) > 0. \end{aligned}$$

In brief, for $k = 1, 2, \cdots, \nu - 1$,

$$\Delta_k \arg f(z) = (\pi/2)[\operatorname{sg} \rho(x_{k+1} - \epsilon) - \operatorname{sg} \rho(x_k + \epsilon)]. \tag{37,7}$$

We shall now compute the changes $\Delta_0 \arg f(z)$ and $\Delta_\nu \arg f(z)$, as $z = x$ varies from $-\infty$ to x_1 and from x_ν to $+\infty$ respectively. Since x_1 and x_ν are the smallest and largest zeros of $P_0(x)$,

$$\operatorname{sg} \rho(x_1 - \epsilon) = \operatorname{sg} \rho(-\infty), \qquad \operatorname{sg} \rho(x_\nu + \epsilon) = \operatorname{sg} \rho(+\infty).$$

This means that

$$(37,8)\qquad \Delta_0 \arg f(z) = (\pi/2)\,\mathrm{sg}\,\rho(x_1 - \epsilon), \qquad \Delta_\nu \arg f(z) = -(\pi/2)\,\mathrm{sg}\,\rho(x_\nu + \epsilon).$$

From (37,7) and (37,8) we may compute the net change $\Delta_L \arg f(x)$ as x varies from $-\infty$ to $+\infty$. This net change is

$$\Delta_L \arg f(z) = \frac{\pi}{2}\left\{ \sum_{k=1}^{\nu-1} [\mathrm{sg}\,\rho(x_{k+1} - \epsilon) - \mathrm{sg}\,\rho(x_k + \epsilon)] \right.$$
$$\left. + \,\mathrm{sg}\,\rho(x_1 - \epsilon) - \mathrm{sg}\,\rho(x_\nu + \epsilon)\right\}.$$

That is,

$$(37,9)\qquad \Delta_L \arg f(z) = \pi \sum_{k=1}^{\nu} \left[\frac{\mathrm{sg}\,\rho(x_k - \epsilon) - \mathrm{sg}\,\rho(x_k + \epsilon)}{2}\right].$$

Defined as the *Cauchy Index* of the function $\rho(x)$ at the point $x = x_k$ [Cauchy 2], the bracket in eq. (37,9) has the values -1, or $+1$ or 0 according as $\rho(x_k - \epsilon) < 0$ and $\rho(x_k + \epsilon) > 0$, $\rho(x_k - \epsilon) > 0$ and $\rho(x_k + \epsilon) < 0$ or $\rho(x_k - \epsilon)\rho(x_k + \epsilon) > 0$. If, therefore, σ is the number of x_k at which, as x increases from $-\infty$ to ∞, $\rho(x)$ changes from $-$ to $+$ and τ the number of x_k at which $\rho(x)$ changes from $+$ to $-$, then eq. (37,9) may be rewritten as

$$(37,10)\qquad \Delta_L \arg f(z) = \pi(\tau - \sigma).$$

In view of eqs. (1,7), (1,8), (1,9) and (37,10) we may state the *Cauchy Index Theorem* essentially as presented in Hurwitz [2].

THEOREM (37,1). *Let*

$$(37,11)\qquad f(z) = a_0 + a_1 z + \cdots + a_{n-1}z^{n-1} + z^n = P_0(z) + iP_1(z)$$

where $P_0(z)$ and $P_1(z)$ are real polynomials with $P_1(z) \not\equiv 0$. As the point $z = x$ moves on the real axis from $-\infty$ to $+\infty$, let σ be the number of real zeros of $P_0(z)$ at which $\rho(x) = P_0(x)/P_1(x)$ changes from $-$ to $+$, and τ the number of real zeros of $P_0(z)$ at which $\rho(x)$ changes from $+$ to $-$. If $f(z)$ has no real zeros, p zeros in the upper half-plane and q zeros in the lower half-plane, then

$$(37,12)\qquad p = (1/2)[n + (\tau - \sigma)], \qquad q = (1/2)[n - (\tau - \sigma)].$$

EXERCISES. Prove the following.

1. All the zeros of the $f(z)$ of Th. (37,1) lie in the upper (lower) half-plane if $P_0(x)$ has n real zeros x_k and if, at each x_k, $\rho'(x_k) < 0$ (> 0).

2. Let $F(z) = A_0 + A_1 z + \cdots + A_{n-1}z^{n-1} + (-i)^n z^n$,

$$P_0(z) = a_0' + a_1'z + \cdots + a_{n-1}'z^{n-1} + z^n,$$

$$P_1(z) = a_0'' + a_1''z + \cdots + a_{n-1}''z^{n-1},$$

where $a_k' = \Re(i^k A_k)$ and $a_k'' = \Im(i^k A_k)$. Let σ be the number of real zeros of $P_0(y)$ at which the ratio $\rho(y) = P_0(y)/P_1(y)$ changes sign from $-$ to $+$ and τ the number of real zeros of $P_0(y)$ at which $\rho(y)$ changes sign from $+$ to $-$, as y varies through real values from $-\infty$ to $+\infty$. Then if $F(z)$ has p zeros in the left half-plane $\Re(z) < 0$ and q zeros in the right half-plane $\Re(z) > 0$ and if $p + q = n$, then p and q are given by eqs. (37,12). Hint: Apply Th. (37,1) to $f(z) = F(iz)$.

3. If in Th. (37,1) $P_0(x)$ has n real zeros x_k and $P_1(x)$ has $n - 1$ real zeros X_k with $x_1 < X_1 < x_2 < X_2 < \cdots < X_{n-1} < x_n$, then $p = 0$ and $q = n$ if $(-1)^n P_1(x_1) < 0$, but $p = n$ and $q = 0$ if $(-1)^n P_1(x_1) > 0$.

4. If $p = q - n = 0$ or $p - n = q = 0$, then for arbitrary real constants A and B the polynomial $AP_0(z) + BP_1(z)$ has n distinct real zeros [Hermite 2; Biehler 1; Laguerre 1, p. 109 and 360; Hurwitz 2; Obrechkoff 6].

5. If $P_0(x)$ has n real zeros x_k with $x_1 < x_2 < \cdots < x_n$ and if $P_1(x) > 0$ for $x_1 < x < x_n$, then $p = m$ and $q = m$ or $q = m + 1$ according as $n = 2m$ or $n = 2m + 1$. If $P_1(x) < 0$ for $x_1 < x < x_n$, the values of p and q are interchanged.

6. If the $f(z)$ of Th. (37,1) has exactly r real zeros $\xi_1, \xi_2, \cdots, \xi_r$ and if in computing σ and τ the sign changes of $\rho(x)$ at the points ξ_k be not included, then

$$p = (1/2)(n - r + \tau - \sigma) \qquad \text{and} \qquad q = (1/2)(n - r - \tau + \sigma).$$

7. If $P(z)$ is an nth degree polynomial and $P^*(z) = \overline{P}(-z)$, then the $(n - 1)$st degree polynomial

$$P_1(z) = [P^*(z_1)P(z) - P(z_1)P^*(z)]/(z - z_1),$$

where $|P^*(z_1)| > |P(z_1)|$, has one less zero than $P(z)$ at points z for which $\Re(z)\Re(z_1) > 0$ and has the same number of zeros as $P(z)$ at points z for which $\Re(z)\Re(z_1) < 0$. Hint: Use Rouché's Theorem (Th. 1,3) to show that $P(z)$ and $P(z) + \lambda P^*(z)$ where $|\lambda| < 1$ have the same number of zeros in both half-planes $\Re(z) > 0$ and $\Re(z) < 0$ [Schur 3, Benjaminowitsch 2, Frank 2].

8. Let $f(z) = \prod_{j=1}^{m} (z - z_j) = z^m + a_1 z^{m-1} + \cdots + a_m$,

$$g(z) = \prod_{j=1}^{m} \prod_{j=k+1}^{m} (z - z_j - z_k) = z^n + b_1 z^{n-1} + \cdots + b_n,$$

where $n = m(m - 1)/2$. If all the a_j are real, all the zeros of $f(z)$ lie in the half-plane $\Re(z) < 0$ if and only if $a_j > 0$ for $j = 1, 2, \cdots, m$ and $b_k > 0$ for $k = 1, 2, \cdots, n$ [Routh 1, 2; Bateman 1].

38. **Sturm sequences.** By Th. (37,1) we have reduced the problem of finding the number of zeros of $f(z)$ in the upper and lower half-planes to the problem of calculating the difference $\tau - \sigma$. In the case of real polynomials this difference has been computed in Hurwitz [2] by use of the theory of residues and quadratic forms and in Routh [1] and [2] by use of Sturm sequences. We shall follow the latter method. (Cf. Serret [1].)

Let us construct the sequence of functions $P_0(x)$, $P_1(x)$, $P_2(x)$, $\cdots$, $P_\mu(x)$ by applying to $P_0(x)$ and $P_1(x)$ in (37,4) the division algorithm in which the remainder is written with a negative sign; viz.,

$$(38,1) \qquad P_{k-1}(x) = Q_k(x)P_k(x) - P_{k+1}(x), \qquad k = 1, 2, \cdots, \mu - 1,$$

where $P_{k-1}(x)$, $P_k(x)$, $P_{k+1}(x)$ and $Q_k(x)$ are polynomials with

$$\deg Q_k(x) = \deg P_{k-1}(x) - \deg P_k(x) > 0 \qquad (\deg \equiv \text{degree of}).$$

The algorithm is continued until, for μ sufficiently large, $P_\mu(x) = Cg(x)$ where C is a constant and $g(x)$ is the greatest common divisor of $P_0(x)$ and $P_1(x)$. If $g(x)$ is not a constant, its zeros are non-real since $f(x) \neq 0$ at points x of the real axis. In any case, therefore, for all real x

$$(38,2) \qquad \operatorname{sg} P_\mu(x) = \text{const.} \neq 0.$$

As x varies from $-\infty$ to $+\infty$, let us consider

$$(38,3) \qquad \mathcal{V}\{P_k(x)\} \equiv \mathcal{V}\{P_0(x), P_1(x), \cdots, P_\mu(x)\},$$

the number of variations of sign in the sequence $P_0(x)$, $P_1(x)$, $\cdots$, $P_\mu(x)$. This number cannot change except possibly at a zero ξ of some $P_k(x)$.

If $0 < k < \mu$, then $P_k(\xi) = 0$ implies according to eq. (38,1) that $P_{k-1}(\xi) = -P_{k+1}(\xi)$. This in turn implies that $P_{k-1}(\xi)P_{k+1}(\xi) < 0$, for otherwise $P_{k-1}(\xi) = P_{k+1}(\xi) = 0$ and consequently $P_j(\xi) = 0$ for all $j > k$ including $j = \mu$, in contradiction to eq. (38,2). In brief, $P_k(\xi) = 0$ with $0 < k < \mu$ does not entail at $x = \xi$ any change in $\mathcal{V}\{P_k(x)\}$.

In the case that $P_0(\xi) = 0$, we have already indicated that for any sufficiently small positive number ϵ, $\operatorname{sg} P_1(\xi) = \operatorname{sg} P_1(\xi - \epsilon) = \operatorname{sg} P_1(\xi + \epsilon) \neq 0$. If also $\operatorname{sg} P_0(\xi - \epsilon) = \operatorname{sg} P_0(\xi + \epsilon)$, no change in $\mathcal{V}\{P_k(x)\}$ occurs at $x = \xi$. If, however,

$$(38,4) \qquad P_0(\xi - \epsilon)P_1(\xi - \epsilon) > 0, \qquad P_0(\xi + \epsilon)P_1(\xi + \epsilon) < 0,$$

then $\mathcal{V}\{P_k(x)\}$ will increase by one at $x = \xi$; whereas, if

$$(38,5) \qquad P_0(\xi - \epsilon)P_1(\xi - \epsilon) < 0, \qquad P_0(\xi + \epsilon)P_1(\xi + \epsilon) > 0,$$

$\mathcal{V}\{P_k(x)\}$ will decrease by one at $x = \xi$. But, as $P_0(x)P_1(x) = \rho(x)[P_1(x)]^2$,

$$(38,6) \qquad \operatorname{sg}[P_0(x)P_1(x)] = \operatorname{sg} \rho(x)$$

in the neighborhood of any zero x_k of $P_0(x)$. In terms of the numbers σ and τ defined in Th. (37,1), ineqs. (38,4) are satisfied by τ zeros x_k of $P_0(x)$ and ineqs. (38,5) are satisfied by σ zeros x_k of $P_0(x)$. This means that

$$(38,7) \qquad \tau - \sigma = \mathcal{V}\{P_k(+\infty)\} - \mathcal{V}\{P_k(-\infty)\}.$$

In view of this result, we may restate Th. (37,1), as

THEOREM (38,1). *Let*

$$(38,8) \qquad f(z) = a_0 + a_1 z + \cdots + a_{n-1}z^{n-1} + z^n = P_0(z) + iP_1(z),$$

where $P_0(z)$ and $P_1(z)$ are real polynomials and $P_1(z) \not\equiv 0$, be a polynomial which has no real zeros, p zeros in the upper half-plane and q zeros in the lower half-plane. Let $P_0(x)$, $P_1(x)$, $\cdots$, $P_\mu(x)$ be the Sturm sequence formed by applying to $P_0(x)/P_1(x)$ the negative-remainder, division algorithm. Then

$$(38,9) \qquad p = (1/2)[n + \mathcal{V}\{P_k(+\infty)\} - \mathcal{V}\{P_k(-\infty)\}],$$

$$(38,10) \qquad q = (1/2)[n - \mathcal{V}\{P_k(+\infty)\} + \mathcal{V}\{P_k(-\infty)\}].$$

In order to compute the right sides of eqs. (38,9) and (38,10), let us write the terms of highest degree in $P_k(x)$ as $b_k x^{m_k}$, $b_k \neq 0$, and that in $Q_k(x)$ as $c_k x^{n_k}$. Clearly, $b_0 = 1$, $b_1 = a''_{m_1}$; $m_0 = n$, $m_1 \leqq n - 1$, $m_2 \leqq n - 2$, $m_3 \leqq n - 3$, $\cdots$, $m_\mu \leqq n - \mu$; $n_1 = 1$, $n_2 = m_1 - m_2$, $n_3 = m_2 - m_3$, $\cdots$, $n_\mu = m_{\mu-1} - m_\mu$. By equating the coefficients of x^{m_k} on both sides of eq. (38,1), we find that

$$(38,11) \qquad c_k = b_{k-1}/b_k \neq 0.$$

Obviously, $\operatorname{sg} P_k(+\infty) = \operatorname{sg} b_k$ and $\operatorname{sg} P(-\infty) = (-1)^{m_k} \operatorname{sg} b_k$. In consequence,

$$\begin{aligned}
&\mathcal{V}\{P_k(+\infty)\} - \mathcal{V}\{P_k(-\infty)\} \\
&\quad = \mathcal{V}\{1, b_1, \cdots, b_\mu\} - \mathcal{V}\{(-1)^n, (-1)^{m_1}b_1, \cdots, b_\mu\} \\
(38,12) \qquad &\quad = \mathcal{V}\{1, c_1, c_1c_2, c_1c_2c_3, \cdots, c_1c_2\cdots c_\mu\} \\
&\qquad -\mathcal{V}\{1, (-1)^{n-m_1}c_1, (-1)^{n-m_2}c_1c_2, \cdots, (-1)^{n-m_\mu}c_1c_2\cdots c_\mu\} \\
&\quad = \mathcal{N}\{c_1, c_2, \cdots, c_\mu\} - \mathcal{N}\{(-1)^{n_1}c_1, (-1)^{n_2}c_2, \cdots, (-1)^{n_\mu}c_\mu\}
\end{aligned}$$

where $\mathcal{N}\{\lambda_1, \lambda_2, \cdots, \lambda_\mu\}$ denotes the number of negative λ_j in the set λ_1, λ_2, $\cdots$, λ_μ.

Of special interest is the case that $\mu = n$ when each $n_k = 1$. In that case, eq. (38,12) becomes

$$\mathcal{V}\{P_k(+\infty)\} - \mathcal{V}\{P_k(-\infty)\} = \mathcal{N}(c_1, c_2, \cdots, c_n) - \mathcal{P}(c_1, c_2, \cdots, c_n)$$

where $\mathcal{P}(\lambda_1, \lambda_2, \cdots, \lambda_n)$ designates the number of positive λ_j. In this case,

$$\mathcal{N}(c_1, c_2, \cdots, c_n) + \mathcal{P}(c_1, c_2, \cdots, c_n) = n = p + q$$

so that Th. (38,1) becomes

COROLLARY (38,1a). *If in Th. (38,1) $\mu = n$ and if c_k denotes the coefficient of the linear term of the quotient $Q_k(x)$ in eq. (38,1), then*

$$(38,13) \qquad p = \mathcal{N}\{c_1, c_2, \cdots, c_n\} \quad \textit{and} \quad q = \mathcal{P}(c_1, c_2, \cdots, c_n).$$

A further simplification in the case $\mu = n$ results from the fact that $Q_k(x) = c_kx + d_k$. This permits us to write eq. (38,1) in the form

$$\frac{P_{k-1}(x)}{P_k(x)} = c_kx + d_k - \frac{P_{k+1}(x)}{P_k(x)}, \qquad k = 1, 2, \cdots, n-1. \tag{38,14}$$

From eqs. (38,14) we may eliminate $P_2(x), P_3(x), \cdots, P_n(x)$ and put the answer in the form

$$\frac{P_1(x)}{P_0(x)} = \cfrac{1}{c_1x + d_1 - \cfrac{1}{c_2x + d_2 - \cfrac{1}{c_3x + d_3 - \cdot \cdot \cdot - \cfrac{1}{c_{n-1}x + d_{n-1} - \cfrac{1}{c_nx + d_n}}}}}. \tag{38,15}$$

Conversely, if $P_1(x)/P_0(x)$ can be expanded in such a continued fraction, then $\mu = n$ in the negative-remainder, division algorithm. Writing the continued fraction (38,15) more compactly we may reformulate Cor. (38,1a) as

Corollary (38,1b). *If for the $P_0(x)$ and $P_1(x)$ of Th.* (38,1) *there exists the continued fraction expansion*

$$\frac{P_1(x)}{P_0(x)} = \frac{1}{(c_1x + d_1)} - \frac{1}{(c_2x + d_2)} - \frac{1}{(c_3x + d_3)} - \cdots - \frac{1}{(c_nx + d_n)} \tag{38,16}$$

where $c_j \neq 0$ for $j = 1, 2, \cdots, n$, then $p = \mathcal{N}(c_1, c_2, \cdots, c_n)$ and $q = \mathcal{P}(c_1, c_2, \cdots, c_n)$.

This result is due to Wall [1] in the case of real polynomials $f(z)$ and to Frank [1] in the case of complex polynomials $f(z)$.

Exercises. Prove the following.

1. All the zeros of $f(z)$ have positive imaginary parts if and only if $\mathcal{V}\{P_k(+\infty)\} - \mathcal{V}\{P_k(-\infty)\} = n$, or if and only if all $c_j < 0$ in eq. (38,15). (Cf. Wall [1] and Frank [1].)

2. If $F(z)$, $P_0(z)$ and $P_1(z)$ are defined as in ex. (37,2), the number p of zeros of $F(z)$ in the half-plane $\Re(z) < 0$ and the number q of zeros of $F(z)$ in the half-plane $\Re(z) > 0$ are given by the eqs. (38,9), (38,10) and (38,13) and Cor. (38,1b).

3. If $F(z) = A_0 + A_1z + \cdots + A_{n-1}z^{n-1} + e^{-n\alpha i}z^n$, $P_0(z) = a_0' + a_1'z + \cdots + a_{n-1}'z^{n-1} + z^n$ where $a_k' = \Re(e^{k\alpha i}A_k)$ and $P_1(z) = a_0'' + a_1''z + \cdots + a_{n-1}''z^{n-1}$ where $a_k'' = \Im(e^{k\alpha i}A_k)$, then the number p of zeros of $F(z)$ in the sector $\alpha < \arg z < \alpha + \pi$ and the number q of zeros of $F(z)$ in the sector $\alpha + \pi <$

$\arg z < \alpha + 2\pi$, if $p + q = n$, are given by eqs. (38,9), (38,10) and (38,13) and Cor. (38,1b).

4. Let the $f(z)$ of Th. (38,1) have exactly r real zeros $\zeta_1, \zeta_2, \cdots, \zeta_r$, let $g(z) = (z - \zeta_1)(z - \zeta_2) \cdots (z - \zeta_r)$ and define $P_0(z) + iP_1(z) = f(z)/g(z)$. Then $p = (1/2)[n - r + \mathcal{V}\{P_k(+\infty)\} - \mathcal{V}\{P_k(-\infty)\}]$ and $q = (1/2)[n - r - \mathcal{V}\{P_k(+\infty)\} + \mathcal{V}\{P_k(-\infty)\}]$.

39. **Determinant sequences.** Continuing the discussion of the case $\mu = n$ treated in Cor. (38,1a) and Cor. (38,1b), we observe that p and q have been expressed as functions of the c_i which in turn we shall now express as functions of the coefficients of $f(z)$ (cf. Frank [1]).

Let us write

$$P_k(x) = b_{n-k,0} + b_{n-k,1}x + \cdots + b_{n-k,n-k}x^{n-k}, \qquad b_{n-k,n-k} \neq 0. \tag{39,1}$$

Comparing (39,1) with eqs. (37,4), we see that

$$b_{n,n} = 1; \qquad b_{n,j} = a'_j; \qquad b_{n-1,j} = a''_j, \qquad j = 0, 1, \cdots, n-1. \tag{39,2}$$

On substitution from eq. (39,1) into eq. (38,1) we obtain the relation

$$\sum_{j=0}^{n-k+1} b_{n-k+1,j}x^j = (c_kx + d_k)\sum_{j=0}^{n-k} b_{n-k,j}x^j - \sum_{j=0}^{n-k-1} b_{n-k-1,j}x^j.$$

Equating corresponding powers of x on both sides leads us to the following system of equations for the c_k.

$$b_{n-k+1,n-k+1} - c_kb_{n-k,n-k} = 0, \tag{39,3}$$

$$b_{n-k+1,n-k} - c_kb_{n-k,n-k-1} - d_kb_{n-k,n-k} = 0, \tag{39,4}$$

$$b_{n-k+1,j} - c_kb_{n-k,j-1} - d_kb_{n-k,j} + b_{n-k-1,j} = 0, \qquad j = 0, 1, \cdots, n-k-1. \tag{39,5}$$

Let us define

$$B_{n-k,j+1} = b_{n-k+1,j+1} - c_kb_{n-k,j}. \tag{39,6}$$

From (39,3), (39,4) and (39,5) it follows that

$$c_k = b_{n-k+1,n-k+1}/b_{n-k,n-k}, \tag{39,7}$$

$$d_k = B_{n-k,n-k}/b_{n-k,n-k}, \tag{39,8}$$

$$b_{n-k-1,j} = -B_{n-k,j} + d_kb_{n-k,j}. \tag{39,9}$$

Let us define the matrix M_{2n-1} with $2n - 1$ rows and columns:

$$\begin{bmatrix} b_{n-1,n-1} & b_{n-1,n-2} & b_{n-1,n-3} & \cdots & b_{n-1,0} & 0 & 0 & \cdots & 0 \\ b_{n,n} & b_{n,n-1} & b_{n,n-2} & \cdots & b_{n,1} & b_{n,0} & 0 & \cdots & 0 \\ 0 & b_{n-1,n-1} & b_{n-1,n-2} & \cdots & b_{n-1,1} & b_{n-1,0} & 0 & \cdots & 0 \\ 0 & b_{n,n} & b_{n,n-1} & \cdots & b_{n,2} & b_{n,1} & b_{n,0} & \cdots & 0 \\ 0 & 0 & b_{n-1,n-1} & \cdots & b_{n-1,2} & b_{n-1,1} & b_{n-1,0} & \cdots & 0 \\ 0 & 0 & b_{n,n} & \cdots & b_{n,3} & b_{n,2} & b_{n,1} & \cdots & 0 \\ . & . & . & \cdots. & . & . & . & \cdots. & \\ . & . & . & \cdots. & . & . & . & \cdots. & \\ 0 & 0 & 0 & \cdots & b_{n-1,n-1} & b_{n-1,n-2} & b_{n-1,n-3} & \cdots & b_{n-1,0} \end{bmatrix}.$$

We shall now show that by a succession of elementary row operations we may reduce M_{2n-1} to the matrix M'_{2n-1} defined as

$$\begin{bmatrix} b_{n-1,n-1} & b_{n-1,n-2} & b_{n-1,n-3} & b_{n-1,n-4} & \cdots & b_{n-1,0} & 0 & 0 & \cdots & 0 \\ 0 & b_{n-1,n-1} & b_{n-1,n-2} & b_{n-1,n-3} & \cdots & b_{n-1,1} & b_{n-1,0} & 0 & \cdots & 0 \\ 0 & 0 & b_{n-2,n-2} & b_{n-2,n-3} & \cdots & b_{n-2,1} & b_{n-2,0} & 0 & \cdots & 0 \\ 0 & 0 & 0 & b_{n-2,n-2} & \cdots & b_{n-2,2} & b_{n-2,1} & b_{n-2,0} & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & b_{n-3,2} & b_{n-3,1} & b_{n-3,0} & \cdots & 0 \\ . & . & . & . & \cdots. & . & . & & \cdots. & \\ . & . & . & . & \cdots. & . & . & & \cdots. & \\ 0 & 0 & 0 & 0 & \cdots. & . & . & & \cdots & b_{0,0} \end{bmatrix}.$$

Let us define the matrices of one row and $2n - 1$ columns:

$$r_{j,k} = [0, 0, \cdots, 0, b_{n-j,n-j}, b_{n-j,n-j-1}, \cdots, b_{n-j,0}, 0, 0, \cdots, 0],$$

$$R_{j,k} = [0, 0, \cdots, 0, B_{n-j,n-j}, B_{n-j,n-j-1}, \cdots, B_{n-j,0}, 0, 0, \cdots, 0],$$

in which the first $k - 1$ and last $n - k + j - 1$ elements are zeros. Then from eqs. (39,6) to (39,9) it follows that

$$r_{j-1,k} - c_j r_{j,k} = R_{j,k+1}, \qquad d_j r_{j,k} - R_{j,k} = r_{j+1,k+1}.$$

We may then write

$$M_{2n-1} = \begin{bmatrix} r_{1,1} \\ r_{0,1} \\ r_{1,2} \\ r_{0,2} \\ \cdot \\ \cdot \\ \cdot \\ r_{1,n-1} \\ r_{0,n-1} \\ r_{1,n} \end{bmatrix}, \qquad M'_{2n-1} = \begin{bmatrix} r_{1,1} \\ r_{1,2} \\ r_{2,3} \\ r_{2,4} \\ \cdot \\ \cdot \\ \cdot \\ r_{n-1,2n-3} \\ r_{n-1,2n-2} \\ r_{n,2n-1} \end{bmatrix}$$

Starting with M_{2n-1}, let us construct the following sequences of matrices by applying the indicated row operations.

$$\begin{bmatrix} r_{1,1} \\ r_{0,1} \\ r_{1,2} \\ r_{0,2} \\ \cdot \\ \cdot \\ \cdot \\ r_{1,n-1} \\ r_{0,n-1} \\ r_{1,n} \end{bmatrix} \rightarrow \begin{bmatrix} r_{1,1} \\ r_{0,1} - c_1 r_{1,1} \\ r_{1,2} \\ r_{0,2} - c_1 r_{1,2} \\ \cdot \\ \cdot \\ \cdot \\ r_{1,n-1} \\ r_{0,n-1} - c_1 r_{1,n-1} \\ r_{1,n} \end{bmatrix} = \begin{bmatrix} r_{1,1} \\ R_{1,2} \\ r_{1,2} \\ R_{1,3} \\ \cdot \\ \cdot \\ \cdot \\ r_{1,n-1} \\ R_{1,n} \\ r_{1,n} \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} r_{1,1} \\ r_{1,2} \\ -R_{1,2} \\ r_{1,3} \\ \cdot \\ \cdot \\ \cdot \\ -R_{1,n-1} \\ r_{1,n} \\ -R_{1,n} \end{bmatrix} \rightarrow \begin{bmatrix} r_{1,1} \\ r_{1,2} \\ d_1r_{1,2} - R_{1,2} \\ r_{1,3} \\ d_1r_{1,3} - R_{1,3} \\ \cdot \\ \cdot \\ \cdot \\ r_{1,n} \\ d_1r_{1,n} - R_{1,n} \end{bmatrix} = \begin{bmatrix} r_{1,1} \\ r_{1,2} \\ r_{2,3} \\ r_{1,3} \\ r_{2,4} \\ \cdot \\ \cdot \\ \cdot \\ r_{1,n} \\ r_{2,n+1} \end{bmatrix}$$

The latter matrix has the same first three rows as M'_{2n-1} and on omission of the first two rows and columns would have the same form as M_{2n-3} . It could therefore be reduced to M'_{2n-1} by repetition of the above row operations.

Let us denote by Δ_k the determinant formed from the first $2k - 1$ elements of the first $2k - 1$ rows and columns of matrix M_{2n-1} and let us denote by Δ'_k the corresponding determinant of matrix M'_{2n-1} . It is well-known that the above operation on the rows of M_{2n-1} make $\Delta'_k = \Delta_k$. Thus

(39,10) $$\Delta_1 = \Delta'_1 = b_{n-1,n-1} ,$$

(39,11) $$\Delta_k = \Delta'_k = b^2_{n-1,n-1}b^2_{n-2,n-2} \cdots b^2_{n-k-1,n-k-1}b_{n-k,n-k}$$

for $k = 2, 3, \cdots , n$. Since $b_{j,j} \neq 0$ for $j = 0, 1, \cdots , n$, it follows then that sg $b_{n-k,n-k} =$ sg Δ_k for $k = 1, 2, \cdots , n$.

Due to eq. (39,7) and due to the relation $b_{n,n} = 1$, we find that

(39,12) $$\mathcal{N}[c_1 , c_2 , \cdots , c_n] = \mathcal{V}[1, b_{n-1,n-1} , b_{n-2,n-2} , \cdots , b_{0,0}],$$

and hence from eqs. (39,10) and (39,11) and Cor. (38,1a) that

(39,13) $$p = \mathcal{V}[1, \Delta_1 , \Delta_2 , \cdots , \Delta_n], \qquad q = \mathcal{V}[1, -\Delta_1 , \Delta_2 , \cdots , (-1)^n\Delta_n].$$

As a summary of the preceding results, let us state

THEOREM (39,1). *Let the coefficients of the polynomial* $f(z) = a_0 + a_1z + \cdots + a_{n-1}z^{n-1} + z^n$ *be written in the form* $a_k = a'_k + ia''_k$, *where* a'_k *and* a''_k *are real. Let* Δ_k *denote the determinant formed from the first* $2k - 1$ *elements in the first* $2k - 1$ *rows and columns of the matrix*

$$\begin{bmatrix} a''_{n-1} & a''_{n-2} & a''_{n-3} & a''_{n-4} & \cdots & a''_0 & 0 & 0 & \cdots & 0 \\ 1 & a'_{n-1} & a'_{n-2} & a'_{n-3} & \cdots & a'_1 & a'_0 & 0 & \cdots & 0 \\ 0 & a''_{n-1} & a''_{n-2} & a''_{n-3} & \cdots & a''_1 & a''_0 & 0 & \cdots & 0 \\ 0 & 1 & a'_{n-1} & a'_{n-2} & \cdots & a'_2 & a'_1 & a'_0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & 0 & \cdots & a''_{n-1} & a''_{n-2} & a''_{n-3} & \cdots & a''_0 \end{bmatrix}.$$

Then if $\Delta_k \neq 0$ for $k = 1, 2, 3, \cdots, n$, the number p of zeros of $f(z)$ in the upper half-plane is equal to the number of variations of sign in the sequence $1, \Delta_1, \Delta_2, \cdots, \Delta_n$, whereas the number q of zeros of $f(z)$ in the lower half-plane is equal to the number of permanences of sign in this sequence.

For the practical purpose of finding the numbers p and q for a given polynomial, the computation of the determinants Δ_k may prove to be quite laborious. Particularly when the computation is to be done by machine, the use of the method leading to the proof may be preferable to the use of the theorem. That is to say, it may be more convenient in such a case to reduce the given matrix M_{2n-1} to the canonical form M'_{2n-1} by successive steps each of which (especially when the a_j are real) can be readily performed on a computing machine. The $b_{k,k}$ needed in eq. (39,12) are the elements in the main diagonal of matrix M'_{2n-1}.

Exercise. Prove the following.

1. Let D_k denote the determinant formed from the first $2k$ elements in the first $2k$ rows and columns of the square matrix $(2n \times 2n)$

$$\begin{bmatrix} 1 & a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_0 & 0 & 0 & \cdots & 0 \\ 1 & \bar{a}_{n-1} & \bar{a}_{n-2} & \bar{a}_{n-3} & \cdots & \bar{a}_0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & a_{n-1} & a_{n-2} & \cdots & a_1 & a_0 & 0 & \cdots & 0 \\ 0 & 1 & \bar{a}_{n-1} & \bar{a}_{n-2} & \cdots & \bar{a}_1 & \bar{a}_0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & a_{n-1} & \cdots & a_2 & a_1 & a_0 & \cdots & 0 \\ 0 & 0 & 1 & \bar{a}_{n-1} & \cdots & \bar{a}_2 & \bar{a}_1 & \bar{a}_0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & 0 & \cdots & \bar{a}_{n-1} & \bar{a}_{n-2} & \bar{a}_{n-3} & \cdots & \bar{a}_0 \end{bmatrix}.$$

Then the Δ_k of Th. (39,1) may be written as $\Delta_k = (i/2)^k D_k$.

40. **The number of zeros with negative real parts.** We now return to the problem which, as we indicated in sec. 36, is of importance in the study of dynamic stability. Given the polynomial

$$F(z) = z^n + (A_1 + iB_1)z^{n-1} + \cdots + (A_n + iB_n)$$

where the A_j and B_j are real numbers we wish to find the number p of its zeros in the half-plane $\Re(z) > 0$ and the number q of its zeros in the half-plane $\Re(z) < 0$. In particular, we wish to find the conditions for q to be n.

Let us form the polynomial

$$f(z) = i^n F(-iz) = a_0 + a_1 z + \cdots + a_{n-1}z^{n-1} + z^n$$

where $a_k = a'_k + ia''_k = i^{n-k}(A_{n-k} + iB_{n-k})$. Thus for $m = 0, 1, 2, \cdots$

$$a'_{n-4m} = A_{4m}, \; a'_{n-4m-1} = -B_{4m+1}, \; a'_{n-4m-2} = -A_{4m+2}, \; a'_{n-4m-3} = B_{4m+3};$$

$$a''_{n-4m} = B_{4m}, \; a''_{n-4m-1} = A_{4m+1}, \; a''_{n-4m-2} = -B_{4m+2}, \; a''_{n-4m-3} = -A_{4m+3}.$$

If we further define

$$A_j = B_j = 0 \qquad \text{for} \qquad j > n,$$

we may write the determinant Δ_k of Th. (39,1) as

$$(40,1) \qquad \begin{vmatrix} A_1, & -B_2, & -A_3, & B_4, & A_5, & \cdots, & (-1)^{k+1} A_{2k-1} \\ 1, & -B_1, & -A_2, & B_3, & A_4, & \cdots, & (-1)^{k+1} A_{2k-2} \\ 0, & A_1, & -B_2, & -A_3, & B_4, & \cdots, & (-1)^{k+1} B_{2k-2} \\ 0, & 1, & -B_1, & -A_2, & B_3, & \cdots, & (-1)^{k+1} B_{2k-3} \\ 0, & 0, & A_1, & -B_2, & -A_3, & \cdots, & (-1)^k A_{2k-3} \\ 0, & 0, & 1, & -B_1, & -A_2, & \cdots, & (-1)^k A_{2k-4} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \end{vmatrix}.$$

By shifting certain rows and columns and changing the signs of certain rows and columns, we may change (40,1) to the form given in Th. (40,1) below.

Since the substitution $-iz$ for z corresponds to a rotation of the plane by an angle $\pi/2$ about $z = 0$, $f(z)$ has p zeros in the upper half plane and q zeros in the lower half plane. According to Th. (39,1), $F(z)$ has therefore as many

zeros in the half planes $\Re(z) > 0$ and $\Re(z) < 0$ as there are variations and permanences respectively in the sequences of the determinants (40,1).

Thus, we have proved

THEOREM (40,1). *Given the polynomial*

$$F(z) = z^n + (A_1 + iB_1)z^{n-1} + \cdots + (A_n + iB_n) \qquad (A_j\,,\, B_j \text{ real}),$$

let us form the determinants $\Delta_1 = A_1$ *and*

$$\Delta_k = \begin{vmatrix} A_1, & A_3, & A_5, & \cdots, & A_{2k-1}, & -B_2, & -B_4, & \cdots, & -B_{2k-2} \\ 1, & A_2, & A_4, & \cdots, & A_{2k-2}, & -B_1, & -B_3, & \cdots, & -B_{2k-3} \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\ 0, & 0, & 0, & \cdots, & A_k, & 0, & 0, & \cdots, & -B_{k-1} \\ 0, & B_2, & B_4, & \cdots, & B_{2k-2}, & A_1, & A_3, & \cdots, & A_{2k-3} \\ 0, & B_1, & B_3, & \cdots, & B_{2k-3}, & 1, & A_2, & \cdots, & A_{2k-4} \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\ 0, & 0, & 0, & \cdots, & B_k, & 0, & 0, & \cdots, & A_{k-1} \end{vmatrix}$$

for $k = 2, 3, \cdots, n$, *with* $A_j = B_j = 0$ *for* $j > n$. *Let us denote by* p *and* q *the number of zeros of* $f(z)$ *in the half-planes* $\Re(z) > 0$ *and* $\Re(z) < 0$. *If* $\Delta_k \neq 0$ *for* $k = 1, 2, \cdots, n$, *then*

$$p = \mathfrak{V}(1, \Delta_1, \Delta_2, \cdots, \Delta_n)$$

and

$$q = \mathfrak{V}(1, -\Delta_1, \Delta_2, \cdots, (-1)^n\Delta_n).$$

In the case $\Delta_k > 0$, $k = 1, 2, \cdots, n$, this theorem is stated explicitly in Frank [1] and in Bilharz [3], the latter with the Δ_k in the form (40,1). The general case may, however, be deduced from other statements in these two papers.

Of special interest is the case that $F(z)$ is a real polynomial. In that case, $B_j = 0$ for all j; $\Delta_1 = \delta_1$ and $\Delta_k = \delta_k\delta_{k-1}$, where δ_k is the determinant defined in Th. (40,2) below.

Since sg $\Delta_1\Delta_2 =$ sg δ_2 and sg $\Delta_k\Delta_{k+1} =$ sg $(\delta_{k-1}\delta_{k+1})$ for $k = 2, 3, \cdots, n - 1$, we may state the following theorem.

THEOREM (40,2). *Given the real polynomial*

$$F(z) = z^n + A_1 z^{n-1} + \cdots + A_n,$$

let us form the determinants $\delta_1 = A_1$ *and*

$$\delta_k = \begin{vmatrix} A_1, & A_3, & A_5, & \cdots, & A_{2k-1} \\ 1, & A_2, & A_4, & \cdots, & A_{2k-2} \\ 0, & A_1, & A_3, & \cdots, & A_{2k-3} \\ 0, & 1, & A_2, & \cdots, & A_{2k-4} \\ \cdot, & \cdot, & \cdot, & \cdots, & \cdot \\ 0, & 0, & 0, & \cdots, & A_k \end{vmatrix}$$

for $k = 2, 3, \cdots, n$, *with* $A_j = 0$ *for* $j > n$. *Let us denote by* p *and* q *the number of zeros of* $F(z)$ *in the half-planes* $\Re(z) > 0$ *and* $\Re(z) < 0$ *respectively. Furthermore let us define* $r = 0$ *or* 1 *according as* n *is even or odd and let us set*

$$\epsilon_{2k-1} = (-1)^k \delta_{2k-1};$$

$$\epsilon_{2k} = (-1)^k \delta_{2k}.$$

If $\delta_k \neq 0$ *for* $k = 1, 2, \cdots, n$, *then*

$$p = \mathfrak{V}(1, \delta_1, \delta_3, \cdots, \delta_{n-1+r}) + \mathfrak{V}(1, \delta_2, \delta_4, \cdots, \delta_{n-r}),$$

$$q = \mathfrak{V}(1, \epsilon_1, \epsilon_3, \cdots, \epsilon_{n-1+r}) + \mathfrak{V}(1, \epsilon_2, \epsilon_4, \cdots, \epsilon_{n-r}).$$

In particular, if $\delta_k > 0$ for all k, then $p = 0$ and $q = n$. This leads us to the well-known result due to Hurwitz [2] which we state as the

HURWITZ CRITERION (Cor. 40,2). *If all the determinants* δ_k *defined in Th.* (40,2) *are positive, the polynomial* $F(z)$ *has only zeros with negative real parts.*

For other proofs of Cor. (40,2) we refer the reader to Bompiani [1], Orlando [1] and [2], Lienard-Chipart [1], Fujiwara [1] and [4], Schur [3], Vahlen [1], Obrechkoff [6], Wall [1], and Neĭmark [1].

Regarding the practical computation of the δ_k, the reader is referred to the remarks following Th. (39,1). Regarding the case that some of the δ_k are zero, the reader is referred to the discussion in sec. 44.

EXERCISE. Prove the following.

1. Th. (40,2) is valid for the number of zeros of the real polynomial

$$\phi(z) = \alpha_0 + \alpha_1 z + \cdots + \alpha_n z^n, \qquad \alpha_0 > 0,$$

in the half-planes $\Re(z) > 0$ and $\Re(z) < 0$, when the δ_k are replaced by the determinants

$$\delta_k' = \begin{vmatrix} \alpha_1 & \alpha_3 & \alpha_5 & \alpha_7 & \cdots & \alpha_{2k-1} \\ \alpha_0 & \alpha_2 & \alpha_4 & \alpha_6 & \cdots & \alpha_{2k-2} \\ 0 & \alpha_1 & \alpha_3 & \alpha_5 & \cdots & \alpha_{2k-3} \\ 0 & \alpha_0 & \alpha_2 & \alpha_4 & \cdots & \alpha_{2k-4} \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & 0 & \cdots & \alpha_k \end{vmatrix}.$$

Hint: Apply Th. (40,2) to $F(z) = (z^n/\alpha_0)\phi(1/z)$.

41. **The number of zeros in a sector.** The right half-plane is the special case $S(\pi/2)$ of the sector $S(\gamma)$ comprised of all points z for which

$$|\arg z| \leqq \gamma < \pi. \tag{41,1}$$

In extension of our previous results on the number of zeros of the polynomial

$$f(z) = a_0 + a_1 z + \cdots + a_n z^n, \qquad a_0 a_n \neq 0, \tag{41,2}$$

in the right half-plane, let us now outline the methods of determining the number of zeros of $f(z)$ in the region $S(\gamma)$.

For this purpose, let us set

$$a_k/a_n = A_k e^{i\alpha_k}, \qquad 0 \leqq \alpha_k < 2\pi, \qquad z = re^{i\theta},$$

and

$$F(z) = f(re^{i\theta})/a_n e^{in\theta} = P_0(r, \theta) + iP_1(r, \theta) \tag{41,3}$$

where

$$\begin{aligned} P_0(r, \theta) = A_0 \cos[\alpha_0 - n\theta] + A_1 r \cos[\alpha_1 - (n-1)\theta] + \cdots \\ + A_{n-1} r^{n-1} \cos[\alpha_{n-1} - \theta] + r^n, \end{aligned} \tag{41,4}$$

$$\begin{aligned} P_1(r, \theta) = A_0 \sin[\alpha_0 - n\theta] + A_1 r \sin[\alpha_1 - (n-1)\theta] + \cdots \\ + A_{n-1} r^{n-1} \sin[\alpha_{n-1} - \theta]. \end{aligned} \tag{41,5}$$

Let us denote by $r_1, r_2, \cdots, r_\mu$, $\mu \leqq n$, the positive zeros of $P_0(r, \gamma)$ and by

$r_1', r_2', \cdots, r_\nu', \nu \leqq n$, the positive zeros of $P_0(r, -\gamma)$, these zeros being labelled so that

$$(41,6) \qquad 0 < r_1 < r_2 < \cdots < r_\mu, \qquad 0 < r_1' < r_2' < \cdots < r_\nu'.$$

Let us assume that

$$(41,7) \qquad P_0(0, \gamma)P_0(0, -\gamma) = A_0^2 \cos(\alpha_0 - n\gamma)\cos(\alpha_0 + n\gamma) \neq 0,$$

which means, since $A_0 \neq 0$, that for any integer m

$$(41,8) \qquad \alpha_0 \pm n\gamma \neq (2m + 1)(\pi/2).$$

In analogy with Th. (37,1) and by the same methods, we may prove the following.

THEOREM (41,1). *Let the polynomial $f(z) = a_0 + a_1z + \cdots + a_nz^n$ have p zeros interior to the sector $S(\gamma)$ and no zeros on the boundary s of this region. Let $\Delta_s \arg f(z)$ be the net change in $\arg f(z)$ as point z traverses s in the positive direction. Then*

$$(41,9) \qquad 2\pi p = 2n\gamma + \Delta_s \arg f(z).$$

In terms of the ratio

$$(41,10) \qquad \rho(r, \theta) = P_0(r, \theta)/P_1(r, \theta)$$

we may derive a formula similar to (37,9); namely,

$$(41,11) \qquad \begin{aligned} 2n\gamma + \Delta_s \arg f(z) = \pi(\kappa + \kappa') &+ \pi \sum_{j=1} \left[\frac{\text{sg}\, \rho(r_j + \epsilon, \gamma) - \text{sg}\, \rho(r_j - \epsilon, \gamma)}{2}\right] \\ &- \pi \sum_{j=1}^{\nu} \left[\frac{\text{sg}\, \rho(r_j' + \epsilon, -\gamma) - \text{sg}\, \rho(r_j' - \epsilon, -\gamma)}{2}\right], \end{aligned}$$

where κ and κ' are integers (or zero) such that

$$|\alpha_0 - n\gamma + \kappa\pi| < \pi/2, \qquad |\alpha_0 + n\gamma - \kappa'\pi| < \pi/2.$$

For, let us note that, since $f(0) = a_0 \neq 0$, also $f(z) \neq 0$ for all $|z| \leqq \epsilon$, ϵ being a sufficiently small positive number. Hence, the number of zeros of $f(z)$ in $S(\gamma)$ will be the same as in the region $S^*(\gamma)$ comprised only of the points of $S(\gamma)$ for which $|z| \geqq \epsilon$. Let us denote by Γ the arc of the circle $|z| = \epsilon$ lying in $S(\gamma)$ and by β the complete boundary of $S^*(\gamma)$. Then

$$\begin{aligned} \Delta_\beta \arg F(z) = \frac{\pi}{2} \sum_{j=1}^{\mu-1} &[-\text{sg}\, \rho(r_{j+1} - \epsilon, \gamma) + \text{sg}\, \rho(r_j + \epsilon, \gamma)] \\ &+ \frac{\pi}{2} \sum_{j=1}^{\nu-1} [+\text{sg}\, \rho(r_{j+1}' - \epsilon, -\gamma) - \text{sg}\, \rho(r_j' + \epsilon, -\gamma)] \\ &+ \Delta_\mu \arg F(z) + \Delta_0 \arg F(z) + \Delta_\Gamma \arg F(z) \\ &+ \Delta_0' \arg F(z) + \Delta_\nu' \arg F(z), \end{aligned}$$

where the last five terms denote respectively the increments in arg $F(z)$ as point z moves along the ray $\theta = \gamma$ from $r = \infty$ to $r = r_\mu$ and from $r = r_1$ to $r = \epsilon$, along Γ, and along the ray $\theta = -\gamma$ from $r = \epsilon$ to $r = r_1'$ and from $r = r_1'$ to $r = \infty$. It is clear that

$$\begin{aligned}
\Delta_\mu \arg F(z) &= (\pi/2) \operatorname{sg} \rho(r_\mu + \epsilon, \gamma), \\
\Delta_0 \arg F(z) &= \alpha_0 - n\gamma + \kappa\pi - (\pi/2) \operatorname{sg} \rho(r_1 - \epsilon, \gamma), \\
\Delta_0' \arg F(z) &= (\pi/2) \operatorname{sg} \rho(r_1' - \epsilon, -\gamma) + \kappa'\pi - \alpha_0 - n\gamma, \\
\Delta_\nu' \arg F(z) &= -(\pi/2) \operatorname{sg} \rho(r_\nu' + \epsilon, -\gamma), \\
\Delta_\Gamma \arg F(z) &= 2n\gamma.
\end{aligned}$$

These formulas lead now to eq. (41,11) since

$$\Delta_\beta \arg f(z) = \Delta_\beta \arg F(z) - 2n\gamma.$$

Let us denote by σ and τ the number of r_j at which, as r increases from 0 to ∞, $\rho(r, \gamma)$ changes from $-$ to $+$ and from $+$ to $-$ respectively and by σ' and τ' the number of r_j' at which, as r increases from 0 to ∞, $\rho(r, -\gamma)$ changes from $-$ to $+$ and from $+$ to $-$ respectively, then (41,9) may be written, due to (41,11), as

$$p = (1/2)[(\sigma - \tau) - (\sigma' - \tau') + (\kappa + \kappa')]. \tag{41,12}$$

Since the differences $(\sigma - \tau)$ and $(\sigma' - \tau')$ may be computed by constructing Sturm sequences, we may state a theorem analogous to Th. (37,2) (cf. Sherman [1] and J. Williams [1]).

THEOREM (41,2). *Let $f(z) = a_0 + a_1 z + \cdots + a_n z^n$ be a polynomial which has p zeros in the sector $\mathcal{S}(\gamma)$ and no zeros on its boundary s. Let $P_0(r, \theta)$ and $P_1(r, \theta)$ be the real polynomials in r such that*

$$f(re^{i\theta})/a_n e^{in\theta} = P_0(r, \theta) + iP_1(r, \theta)$$

and let $P_2(r, \theta), P_3(r, \theta), \cdots, P_\mu(r, \theta) \equiv K(\theta)$ be the Sturm sequence in r obtained by applying the negative-remainder, division algorithm to $P_0(r, \theta)/P_1(r, \theta)$. Finally, let the number of variations in sign in the sequence $P_0(r, \theta), P_1(r, \theta), \cdots, P_\mu(r, \theta)$ be denoted by $V(r, \theta)$. Then

$$\begin{aligned} p = (1/2)\{&[V(0, \gamma) - V(\infty, \gamma)] \\ &- [V(0, -\gamma) - V(\infty, -\gamma)] + \kappa + \kappa'\}. \end{aligned} \tag{41,13}$$

Th. (41,2) is a generalization of the Sturm theorem giving the exact number of zeros of a real polynomial

$$f(z) = a_0 + a_1 z + \cdots + a_n z^n$$

on a given interval of the real axis. This suggests that we also attempt to generalize Descartes' Rule of Signs to complex polynomials.

Before we can proceed to generalize this Rule, we must first formulate a suitable generalization of the concept of the number of variations of sign in a sequence of numbers a_i to cover the case that the a_i are complex numbers.

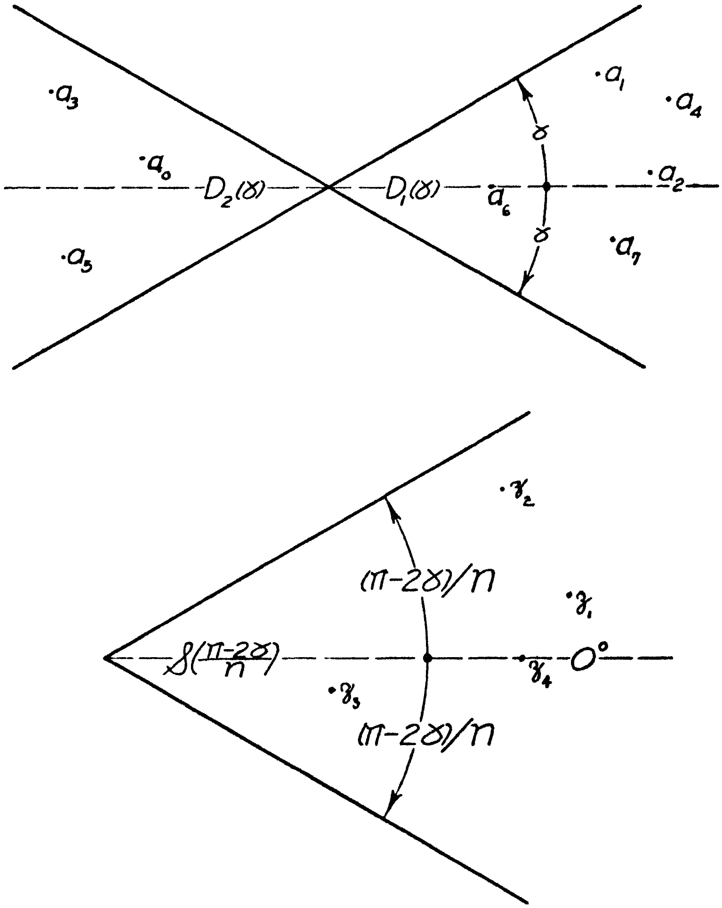

Fig. (41, 1)

Let us denote by $\mathfrak{D}(\gamma)$ the double sector consisting of the two sectors (see fig. 41,1)

$$D_1(\gamma): \quad -\gamma \leqq \arg z \leqq \gamma < \pi/2, \tag{41,14}$$

$$D_2(\gamma): \quad \pi - \gamma \leqq \arg z \leqq \pi + \gamma. \tag{41,15}$$

We shall say as in Schoenberg [1] that a variation with respect to $\mathfrak{D}(\gamma)$ has occurred between a_k and a_{k+1} if a_k lies in one of the sectors $D_1(\gamma)$ or $D_2(\gamma)$ and a_{k+1} lies in the other sector.

This concept permits us to state the following result due to Obrechkoff [2] in the case $\gamma = 0$ and to Schoenberg [1] when $0 \leqq \gamma < \pi/2$.

THEOREM (41,3). *If all the coefficients a_j of the polynomial $f(z) = a_0 + a_1 z + \cdots + a_n z^n$ lie in the double sector $\mathfrak{D}(\gamma)$, then in the sector $\mathrm{S}(\psi)$ defined by the inequality*

$$(41,16) \qquad | \arg z | \leqq \psi < (\pi - 2\gamma)/n$$

the zeros of $f(z)$ number at most $\mathfrak{V}(a_0, a_1, \cdots, a_n)$, the number of variations with respect to $\mathfrak{D}(\gamma)$ in the sequence $a_0, a_1, \cdots, a_n$.

To prove Th. (41,3) let us set $z = re^{i\theta}$, $a_k = A_k e^{i\alpha_k}$ for $k = 0, 1, \cdots, n$ and

$$(41,17) \qquad f(re^{i\theta})e^{-n\theta/2} = Q_0(r, \theta) + iQ_1(r, \theta).$$

If σ, τ, σ' and τ' have the same meaning as above with now $\rho(r, \theta) = Q_0(r, \theta)/Q_1(r, \theta)$, then the above reasoning leads again to eq. (41,12) for the number of zeros of $f(z)$ in the sector (41,16). In particular we infer from (41,12) that, since $\kappa = \kappa' = 0$ here,

$$(41,18) \qquad p \leqq (1/2)(\sigma + \tau + \sigma' + \tau') = (1/2)(m + m'),$$

where m and m' denote the number of positive real zeros of $Q_0(r, \gamma)$ and $Q_0(r, -\gamma)$ respectively.

On the other hand, from eqs. (41,2) and (41,17), we have that

$$Q_0(r, \theta) = \sum_{j=0}^{n} A_j r^j \cos \{\alpha_j - [(n/2) - j]\theta\}.$$

Let us assume that

$$| \alpha_j | \leqq \gamma \leqq \pi/2, \qquad j = 0, 1, \cdots, n,$$

and thus that point a_j lies in $D_1(\gamma)$ or $D_2(\gamma)$ according as $A_j > 0$ or $A_j < 0$. Now on the boundary rays $\theta = \pm\psi$ of the sector (41,16),

$$-\pi/2 < -\gamma - (n/2)\psi \leqq \alpha_j - [(n/2) - j]\theta \leqq \gamma + (n/2)\psi < \pi/2$$

and hence $\cos \{\alpha_j - [(n/2) - j]\theta\} > 0$ for $\theta = \pm\gamma$ and for all j. Applying Descartes' Rule of Sign to $Q_0(r, \gamma)$ and $Q_0(r, -\gamma)$, which are real polynomials in r, we learn that both

$$m \leqq \mathfrak{V}(A_0, A_1, \cdots, A_n), \qquad m' \leqq \mathfrak{V}(A_0, A_1, \cdots, A_n).$$

We now conclude from (41,18) that

$$(41,19) \qquad p \leqq \mathfrak{V}(A_0, A_1, \cdots, A_n),$$

as required in Th. (41,3).

In this extension of Descartes' Rule to complex variables, it is not as yet known whether or not the difference between the right and left sides of ineq. (41,19) is zero or an even integer as it is in the case of real polynomials.

EXERCISES. Prove the following.

1. If all the coefficients of $f(z) = a_0 + a_1 z + \cdots + a_n z^n$ lie in the sector $|\arg z| < \pi/2$, then $f(z)$ has no real positive zeros. More generally, if all the points $z = a_k e^{ik\omega}$ lie in the same convex sector, then $f(z) \neq 0$ on the ray $\theta = \omega$. Hint: Use Th. (1,1) [Kempner 5].

2. If in Th. (41,3) all the a_j are points of the double sector

$$\delta \leqq \arg (\pm z) \leqq \delta + \psi < \delta + \pi,$$

and if $\mathfrak{V}$ is the number of variations of the a_j with respect to this double sector, then $f(z)$ has in the sector $S((\pi - 2\psi)/n)$ at most $\mathfrak{V}$ zeros. Hint: Apply Th. (41,3) to $\{f(z) \exp [-(\delta + \psi/2)i]\}$ [Schoenberg 1].

3. If all the zeros z_k of $f(z) = a_0 + a_1 z + \cdots + a_n z^n$ lie in the sector $A : \delta \leqq \arg z \leqq \delta + \psi < \delta + \pi$, then the points $(-\overline{a_k/a_{k+1}}) = b_k$, $k = 0, 1, \cdots, n-1$, also lie in A. Hint: $\sum_1^n (1/\bar{z}_k) = 1/\bar{b}_0$. According to the proof of Th. (1,1), $1/\bar{b}_0$, as a sum of vectors each of which lies in A, also lies in A and hence b_0 also lies in A. Similarly, express $1/\bar{b}_k$ in terms of the zeros of the kth derivative of $f(z)$ and use Th. (6,1) [Takahashi 1].

CHAPTER X

THE NUMBER OF ZEROS IN A GIVEN CIRCLE

42. **An algorithm.** Let us denote by p ($p \leqq n$) the number of zeros which the polynomial

$$(42,1) \qquad f(z) = a_0 + a_1 z + \cdots + a_n z^n = a_n \prod_{j=1}^{n} (z - z_j)$$

has in a given circle, which, without loss of generality, may be taken as the unit circle $|z| = 1$. One way to determine p would be to map the interior of the unit circle $|z| < 1$ upon the left half w-plane by means of the transformation

$$(42,2) \qquad w = (z-1)/(z+1), \qquad z = (1+w)/(1-w).$$

Then p becomes the number of zeros which the transformed polynomial

$$(42,3) \qquad F(w) = (1-w)^n f((1+w)/(1-w))$$

has in left half-plane and so may be found by applying to $F(w)$ the theorems of Chapter IX. (Cf. Frank [1b].) The result thus obtained appears, however, less elegant than that which we shall presently derive by applying Rouché's Theorem (Th. 1,3) directly to $f(z)$.

Let us associate with $f(z)$ the polynomial

$$(42,4) \qquad f^*(z) = z^n \bar{f}(1/z) = \bar{a}_0 z^n + \bar{a}_1 z^{n-1} + \cdots + \bar{a}_n = \bar{a}_0 \prod_{j=1}^{n} (z - z_j^*)$$

whose zeros $z_k^* = 1/\bar{z}_k$ are, relative to circle $|z| = 1$, the inverses of the zeros z_k of $f(z)$. This means that any zero of $f(z)$ on the unit circle is also a zero of $f^*(z)$ and that, if $f(z)$ has no zeros on the circle $|z| = 1$, then $f^*(z)$ has also no zeros on the circle $|z| = 1$ and has $n - p$ zeros in this circle. Furthermore, on the unit circle, the value of $f^*(z)$ is

$$(42,5) \quad f^*(e^{i\theta}) = \bar{a}_0 \prod_{j=1}^{n} (e^{i\theta} - 1/\bar{z}_j) = \frac{\bar{a}_0 e^{in\theta}(-1)^n}{\bar{z}_1 \bar{z}_2 \cdots \bar{z}_n} \prod_{j=1}^{n} (e^{-i\theta} - \bar{z}_j) = e^{in\theta} \bar{f}(e^{-i\theta})$$

and, consequently,

$$(42,6) \qquad |f^*(e^{i\theta})| = |f(e^{i\theta})|.$$

From $f(z)$ and $f^*(z)$ let us construct the sequence of polynomials $f_j(z) = \sum_{k=0}^{n-j} a_k^{(j)} z^k$, where $f_0(z) = f(z)$ and

$$(42,7) \qquad f_{j+1}(z) = \bar{a}_0^{(j)} f_j(z) - a_{n-j}^{(j)} f_j^*(z), \qquad j = 0, 1, \cdots, n-1.$$

Thus,

$$(42,8) \qquad a_k^{(j+1)} = \bar{a}_0^{(j)} a_k^{(j)} - a_{n-j}^{(j)} \bar{a}_{n-j-k}^{(j)}.$$

In each polynomial $f_j(z)$, the constant term $a_0^{(j)}$ is a real number which we shall denote by δ_j ; viz.,

$$(42,9) \qquad \delta_{j+1} = |\, a_0^{(j)} \,|^2 - |\, a_{n-j}^{(j)} \,|^2 = a_0^{(j+1)}, \qquad j = 0, 1, 2, \cdots, n-1.$$

As to the zeros of these polynomials, Cohn [1] has proved two lemmas which we shall combine in the compact form due to Marden [16].

LEMMA (42,1). *If $f_j(z)$ has p_j zeros in the unit circle C and has no zeros on C and if $\delta_{j+1} \neq 0$, then $f_{j+1}(z)$ has*

$$(42,10) \qquad p_{j+1} = (1/2)\{n - j - [(n-j) - 2p_j] \text{ sg } \delta_{j+1}\}$$

zeros in C and has no zeros on C.

To prove this lemma, let us begin with the case that $\delta_{j+1} > 0$. From eq. (42,6) with $f(z)$ replaced by $f_j(z)$ and from eq. (42,9), we infer that

$$(42,11) \qquad |\, a_{n-j}^{(j)} f_j^*(e^{i\theta}) \,| < |\, a_0^{(j)} f_j(e^{i\theta}) \,|$$

and therefore from Rouché's Theorem that the polynomial $f_{j+1}(z)$ has in C the same number p_j of zeros as $\bar{a}_0^{(j)} f_j(z)$. Since sg $\delta_{j+1} = 1$, this number is in agreement with formula (42,10). Furthermore, ineq. (42,11) makes it impossible for $f_{j+1}(z)$ to have any zeros on the unit circle.

Let us next take the case that $\delta_{j+1} < 0$. Since now

$$(42,12) \qquad |\, a_0^{(j)} f_j(e^{i\theta}) \,| < |\, a_{n-j}^{(j)} f_j^*(e^{i\theta}) \,|,$$

the same reasoning as in the previous case here shows that the polynomial $f_{j+1}(z)$ has in C the same number $(n - j - p_j)$ of zeros as $a_{n-j}^{(j)} f_j^*(z)$. Since now sg $\delta_{j+1} = -1$, this number is also in agreement with formula (42,10). Likewise, ineq. (42,12) makes it impossible for $f_{j+1}(z)$ to have any zeros on the circle C.

Thus, we have proved that Lemma (42,1) is valid in both cases.

Let us now apply Lemma (42,1) to each $f_j(z)$ in the sequence (42,7). We learn thereby that

$$p_1 = (1/2)\{n - (n - 2p) \text{ sg } \delta_1\},$$

$$\begin{aligned} p_2 &= (1/2)\{(n-1) - [n - 1 - 2p_1] \text{ sg } \delta_2\} \\ &= (1/2)\{(n-1) - [(n-1) - n + (n-2p) \text{ sg } \delta_1] \text{ sg } \delta_2\} \\ &= (1/2)\{(n-1) - (n-2p) \text{ sg } (\delta_1\delta_2) + \text{sg } \delta_2\}. \end{aligned}$$

The expression for p_2 is the special case of the formula

$$(42,13) \qquad \begin{aligned} p_j = (1/2)[(n - j + 1) &- (n - 2p) \text{ sg } (\delta_1\delta_2 \cdots \delta_j) \\ &+ \text{sg } (\delta_2\delta_3 \cdots \delta_j) + \text{sg } (\delta_3\delta_4 \cdots \delta_j) + \cdots + \text{sg } \delta_j]. \end{aligned}$$

Let us assume that we have verified formula (42,13) also for $j = 3, 4, \cdots,$

$k - 1$ and on that basis let us compute p_k. From eqs. (42,10) and (42,13) with $j = k - 1$, we obtain

$$p_k = (1/2)\{n - k + 1 - [(n - k + 1) - 2p_{k-1}] \operatorname{sg} \delta_k\}$$

$$= (1/2)\{(n - k + 1) - [(n - k + 1) - (n - k + 2)$$

$$+ (n - 2p) \operatorname{sg} (\delta_1\delta_2 \cdots \delta_{k-1}) - \operatorname{sg} (\delta_2\delta_3 \cdots \delta_{k-1})$$

$$- \operatorname{sg} (\delta_3\delta_4 \cdots \delta_{k-1}) - \cdots - \operatorname{sg} \delta_{k-1}] \operatorname{sg} \delta_k\}$$

$$= (1/2)\{(n - k + 1) - (n - 2p) \operatorname{sg} (\delta_1\delta_2 \cdots \delta_k) + \operatorname{sg} (\delta_2\delta_3 \cdots \delta_k)$$

$$+ \operatorname{sg} (\delta_3\delta_4 \cdots \delta_k) + \cdots + \operatorname{sg} \delta_k\}.$$

This shows by mathematical induction that formula (42,13) holds for all j, $2 \leqq j \leqq n$.

In particular, since $f_n(z) \equiv$ const. and hence $p_n = 0$, we derive from eq. (42,13) with $j = n$ the relation

$$\begin{aligned} &1 - (n - 2p) \operatorname{sg} (\delta_1\delta_2 \cdots \delta_n) + \operatorname{sg} (\delta_2\delta_3 \cdots \delta_n) \\ &+ \operatorname{sg} (\delta_3\delta_4 \cdots \delta_n) + \cdots + \operatorname{sg} \delta_n = 0. \end{aligned} \tag{42,14}$$

If solved for p, eq. (42,14) yields the value

$$p = (1/2)\,(n - \sum_1^n \operatorname{sg} P_k) \tag{42,15}$$

where

$$P_k = \delta_1\delta_2 \cdots \delta_k, \qquad k = 1, 2, \cdots, n. \tag{42,16}$$

To interpret formula (42,15), let us denote by ν the number of negative P_k, $k = 1, 2, \cdots, n$. Then, as $(n - \nu)$ is the number of positive P_k, we may write (42,15) as

$$p = (1/2)[n + \nu - (n - \nu)] = \nu.$$

In other words, we have now as in Marden [16] established

THEOREM (42,1). *If for the polynomial*

$$f(z) = a_0 + a_1 z + \cdots + a_n z^n$$

p of the products P_k defined by eq. (42,16) *are negative and the remaining $n - p$ are positive, then $f(z)$ has p zeros in the unit circle $|z| = 1$, no zeros on this circle and $n - p$ zeros outside this circle.*

We observe that Th. (42,1) does not contain a hypothesis that $f(z) \neq 0$ for $|z| = 1$. This is implied in the hypothesis $\delta_n \neq 0$, as will be seen in secs. 44 and 45.

A convenient way to find the δ_k is by construction of the matrix

$$\begin{bmatrix} a_0 & a_1 & \cdots & a_{n-2} & a_{n-1} & a_n \\ \bar{a}_n & \bar{a}_{n-1} & \cdots & \bar{a}_2 & \bar{a}_1 & \bar{a}_0 \\ a_0^{(1)} & a_1^{(1)} & \cdots & a_{n-2}^{(1)} & a_{n-1}^{(1)} & 0 \\ \bar{a}_{n-1}^{(1)} & \bar{a}_{n-2}^{(1)} & \cdots & \bar{a}_1^{(1)} & \bar{a}_0^{(1)} & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ a_0^{(n)} & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}$$

comprised of the $2n + 1$ rows ρ_j, $j = 1, 2, \cdots, 2n + 1$. Row ρ_1 consists of the coefficients in $f(z) = a_0 + a_1z + \cdots + a_nz^n$ and row ρ_2 consists of the conjugate imaginaries of these coefficients written in the reverse order. In general

$$\rho_{2k+1} = \bar{a}_0^{(k)}\rho_{2k-1} - a_{n-k}^{(k)}\rho_{2k}, \qquad k = 1, 2, \cdots, n,$$

and row ρ_{2k+2} consists of the conjugate imaginaries of the coefficients of the row ρ_{2k+1} written in the reverse order. Since by definition $\delta_k = a_0^{(k)}$, the δ_k are the first elements in the rows ρ_{2k+1}, $k = 1, 2, \cdots, n$.

Exercises. Prove the following.

1. If $\delta_j > 0$ for $j = 2, 3, \cdots, n$, then $f(z) \neq 0$ in $|z| \leqq 1$ or $|z| \geqq 1$ according as $\delta_1 > 0$ or $\delta_1 < 0$. If sg $\delta_j = (-1)^j$, then $p = 2m + 1$ if $n = 4m + 1$ and $p = 2m + 2$ if $n = 4m + k$, $k = 2, 3$ or 4 and $m = 0, 1, 2, \cdots$.

2. Let $\delta_j(r)$ be the values of the δ_j of Th. (42,1) for $a_k = b_kr^k$ and $p(r)$ the corresponding value of p. Then $p(r)$ is the number of zeros of the polynomial $g(z) = b_0 + b_1z + \cdots + b_nz^n$ in the circle $|z| < r$.

3. Let $\delta_j(r, s)$ be the values of the δ_j of Th. (42,1) for $a_j = r^j \sum_{k=j}^{n} C(k, j)s^{k-j}b_k$ and $p(r, s)$ the corresponding value of p. Then $p(r, s)$ is the number of zeros of $g(z) = b_0 + b_1z + \cdots + b_nz^n$ in the circle $|z - s| < r$.

4. If the a_j are all real and if $0 < a_n < a_{n-1} < \cdots < a_0$, then also all the $a_j^{(k)}$ in eq. (42,8) are real and $0 < a_{n-k}^{(k)} < a_{n-k-1}^{(k)} < \cdots < a_0^{(k)}$, for $k = 1, 2, \cdots, n$, and thus $f(z) \neq 0$ in $|z| \leqq 1$ [Eneström 1, Kakeya 1, Cohn 1].

5. If $\delta_j \neq 0$ for $j = 1, 2, \cdots, n - 1$ but if $\delta_n = 0$, then the zero of the $f_{n-1}(z)$ (see eqs. (42,7)) lies on the circle $|z| = 1$.

6. If $|a_0| < |a_n|$, $f(z)$ has all its zeros in the unit circle if and only if $f_1^*(z)$ has all its zeros in the unit circle [Schur 2].

43. Determinant sequences. While the algorithm given in sec. 42 does enable us to find the number p of zeros of the polynomial $f(z)$ in the unit circle, a set of conditions expressed directly in terms of the coefficients of $f(z)$ is de-

sirable, at least from a theoretical standpoint. This type of condition is embodied in the following theorem.

SCHUR-COHN CRITERION (Th. (43,1)). *If for the polynomial* $f(z) = a_0 + a_1z + \cdots + a_nz^n$ *all the determinants*

$$\Delta_k = \begin{vmatrix} a_0 & 0 & 0 & \cdots 0 & a_n & a_{n-1} & \cdots & a_{n-k+1} \\ a_1 & a_0 & 0 & \cdots 0 & 0 & a_n & \cdots & a_{n-k+2} \\ \cdot & \cdot & \cdot & \cdots \cdot & \cdot & \cdot & \cdots & \cdot \\ a_{k-1} & a_{k-2} & a_{k-3} & \cdots a_0 & 0 & 0 & \cdots & a_n \\ \bar{a}_n & 0 & 0 & \cdots 0 & \bar{a}_0 & \bar{a}_1 & \cdots & \bar{a}_{k-1} \\ \bar{a}_{n-1} & \bar{a}_n & 0 & \cdots 0 & 0 & \bar{a}_0 & \cdots & \bar{a}_{k-2} \\ \cdot & \cdot & \cdot & \cdots \cdot & \cdot & \cdot & \cdots & \cdot \\ \bar{a}_{n-k+1} & \bar{a}_{n-k+2} & \bar{a}_{n-k+3} & \cdots \bar{a}_n & 0 & 0 & \cdots & \bar{a}_0 \end{vmatrix}, \quad k = 1, 2, \cdots, n,$$

are different from zero, then $f(z)$ *has no zeros on the circle* $|z| = 1$ *and* p *zeros in this circle,* p *being the number of variations of sign in the sequence* $1, \Delta_1, \Delta_2, \cdots, \Delta_n$.

Th. (43,1) is due to Schur [2] in the case $\Delta_k > 0$ all k and essentially to Cohn [1] in the general case. We shall follow the derivation in Marden [16].

In order to prove Th. (43,1), we need to express the Δ_k in terms of the δ_k entering in Th. (42,1). For this purpose, we shall first develop a reduction formula for the determinants:

$$\Delta_k^{(j)} = \begin{vmatrix} a_0^{(j)} & 0 & 0 & \cdots 0 & a_{n-j}^{(j)} & a_{n-j-1}^{(j)} & \cdots & a_{n-j-k+1}^{(j)} \\ a_1^{(j)} & a_0^{(j)} & 0 & \cdots 0 & 0 & a_{n-j}^{(j)} & \cdots & a_{n-j-k+2}^{(j)} \\ \cdot & \cdot & \cdot & \cdots \cdot & \cdot & \cdot & \cdots & \cdot \\ a_{k-1}^{(j)} & a_{k-2}^{(j)} & a_{k-3}^{(j)} & \cdots a_0^{(j)} & 0 & 0 & \cdots & a_{n-j}^{(j)} \\ \bar{a}_{n-j}^{(j)} & 0 & 0 & \cdots 0 & \bar{a}_0^{(j)} & \bar{a}_1^{(j)} & \cdots & \bar{a}_{k-1}^{(j)} \\ \bar{a}_{n-j-1}^{(j)} & \bar{a}_{n-j}^{(j)} & 0 & \cdots 0 & 0 & \bar{a}_0^{(j)} & \cdots & \bar{a}_{k-2}^{(j)} \\ \cdot & \cdot & \cdot & \cdots \cdot & \cdot & \cdot & \cdots & \cdot \\ \bar{a}_{n-j-k+1}^{(j)} & \bar{a}_{n-j-k+2}^{(j)} & \bar{a}_{n-j-k+3}^{(j)} & \cdots \bar{a}_{n-j}^{(j)} & 0 & 0 & \cdots & \bar{a}_0^{(j)} \end{vmatrix}$$

where the $a_k^{(j)}$ are the quantities defined in eq. (42,8).

With this in mind, let us introduce the determinant of order $2k$

$$\lambda_k^{(j)} = \left| \begin{array}{cccc|cccc} \bar{a}_0^{(j)} & 0 & 0 \cdots & 0 & -a_{n-j}^{(j)} & 0 & 0 \cdots & 0 \\ 0 & \bar{a}_0^{(j)} & 0 \cdots & 0 & 0 & -a_{n-j}^{(j)} & 0 \cdots & 0 \\ \cdot & \cdot & \cdot\cdots & \cdot & \cdot & \cdot & \cdot\cdots & \cdot \\ 0 & 0 & 0 \cdots & \bar{a}_0^{(j)} & 0 & 0 & 0 \cdots & -a_{n-j}^{(j)} \\ \hline -\bar{a}_{n-j}^{(j)} & 0 & 0 \cdots & 0 & a_0^{(j)} & 0 & 0 \cdots & 0 \\ 0 & -\bar{a}_{n-j}^{(j)} & 0 \cdots & 0 & 0 & a_0^{(j)} & 0 \cdots & 0 \\ \cdot & \cdot & \cdot\cdots & \cdot & \cdot & \cdot & \cdot\cdots & \cdot \\ 0 & 0 & 0 \cdots & -\bar{a}_{n-j}^{(j)} & 0 & 0 & 0 \cdots & a_0^{(j)} \end{array} \right| .$$

To evaluate $\lambda_k^{(j)}$, let us multiply its last k rows by $a_{n-j}^{(j)}$ and add the resulting rows to the first k rows multiplied by $a_0^{(j)}$. Using eqs. (42,8), we thus find

$$\lambda_k^{(j)} = (a_0^{(j+1)})^k. \tag{43,1}$$

Let us now form the product $\lambda_k^{(j)}\Delta_k^{(j)}$, which is by the laws of determinant multiplication and by eqs. (42,8):

$$\left| \begin{array}{cccccccc} a_0^{(j+1)} & 0 & 0 & \cdots 0 & 0 & a_{n-j-1}^{(j+1)} & \cdots & a_{n-j-k+1}^{(j+1)} \\ a_1^{(j+1)} & a_0^{(j+1)} & 0 & \cdots 0 & 0 & 0 & \cdots & a_{n-j-k+2}^{(j+1)} \\ \cdot & \cdot & \cdot & \cdots\cdot & \cdot & \cdot & \cdots & \cdot \\ a_{k-1}^{(j+1)} & a_{k-2}^{(j+1)} & a_{k-3}^{(j+1)} & \cdots a_0^{(j+1)} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots 0 & \bar{a}_0^{(j+1)} & \bar{a}_1^{(j+1)} & \cdots & \bar{a}_{k-1}^{(j+1)} \\ \bar{a}_{n-j-1}^{(j+1)} & 0 & 0 & \cdots 0 & 0 & \bar{a}_0^{(j+1)} & \cdots & a_{k-2}^{(j+1)} \\ \cdot & \cdot & \cdot & \cdots\cdot & \cdot & \cdot & \cdots & \cdot \\ \bar{a}_{n-j-k+1}^{(j+1)} & \bar{a}_{n-j-k+2}^{(j+1)} & \bar{a}_{n-j-k+3}^{(j+1)} & \cdots 0 & 0 & 0 & \cdots & \bar{a}_0^{(j+1)} \end{array} \right| .$$

Developing this determinant with respect to the kth and $(k + 1)$st columns according to the Laplace method, we obtain the result

$$\lambda_k^{(j)}\Delta_k^{(j)} = a_0^{(j+1)}\bar{a}_0^{(j+1)}\Delta_{k-1}^{(j+1)}. \tag{43,2}$$

If now we use eqs. (43,1), (43,2) and (42,9), we are led to the following conclusion.

LEMMA (43,1). *The determinants $\Delta_k^{(j)}$ satisfy the relation*

$$\Delta_k^{(j)} = \Delta_{k-1}^{(j+1)}/(\delta_{j+1})^{k-2}. \tag{43,3}$$

Let us now apply Lemma (43,1) to the determinant Δ_k in Th. (43,1), bearing in mind that $a_j \equiv a_j^{(0)}$. Thus, by iteration of (43,3), we have

$$\Delta_k = \frac{\Delta_{k-1}^{(1)}}{\delta_1^{k-2}} = \frac{1}{\delta_1^{k-2}}\frac{1}{\delta_2^{k-3}}\Delta_{k-2}^{(2)} .$$

This suggests the formula

$$\Delta_k = \frac{1}{\delta_1^{k-2}}\frac{1}{\delta_2^{k-3}}\cdots\frac{1}{\delta_{k-1}^{0}}\Delta_1^{(k-1)} = \frac{\delta_k}{\delta_1^{k-2}\delta_2^{k-3}\cdots\delta_{k-2}}$$

which may be established by mathematical induction.

In virtue of this formula,

$$\frac{\Delta_k}{\Delta_{k+1}} = \frac{\delta_k}{\delta_1^{k-2}\delta_2^{k-3}\cdots\delta_{k-2}}\,\frac{\delta_1^{k-1}\delta_2^{k-2}\cdots\delta_{k-1}}{\delta_{k+1}} = \frac{\delta_1\delta_2\cdots\delta_{k+1}}{\delta_{k+1}^2}.$$

This means that

$$\operatorname{sg}(\Delta_k\Delta_{k+1}) = \operatorname{sg} P_{k+1} . \tag{43,4}$$

If now we apply Th. (42,1) in conjunction with eq. (43,4), we may complete the proof of Th. (43,1).

Th. (43,1) may also be proved by using either of the two equivalent Hermitian forms

$$H_1 = \sum_{j=1}^{n} |\,\bar{a}_n u_j + \bar{a}_{n-1}u_{j+1} + \cdots + \bar{a}_j u_n\,|^2$$
$$- \sum_{j=1}^{n} |\,a_0 u_j + a_1 u_{j+1} + \cdots + a_{n-j}u_n\,|^2,$$

$$H_2 = \sum_{j,k=0}^{n-1} A_{jk}u_j\bar{u}_k ,$$

where A_{jk} are the coefficients in the *Bezout resultant*

$$B(f, f^*) = \frac{f(z)f^*(w) - f(w)f^*(z)}{z - w} = \sum_{j,k=0}^{n-1} A_{jk}z^j w^{n-1-k}.$$

Form H_1 was used in Schur [2] and Cohn [1] and form H_2 was used in Fujiwara [5]. If H_1 or H_2 is reduced to the canonical form of a sum of n positive and negative squares, the p and q of Th. (43,1) are respectively the number of positive squares and the number of negative squares. This method is analogous to that which had been previously used in Hermite [1] to determine the number of zeros of a polynomial in the half-planes $\Im(z) > 0$ and $\Im(z) < 0$.

EXERCISES. Prove the following.

1. With the polynomial $f(z) = a_0 + a_1 z + \cdots + a_n z^n$ let there be associated the triangular matrices

$$A_k = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{k-1} \\ 0 & a_0 & a_1 & \cdots & a_{k-2} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & a_0 \end{bmatrix}, \qquad k = 1, 2, \cdots, n.$$

Let $\bar{A}_k$ and A_k^* denote the corresponding matrices for $\bar{f}(z)$ and $f^*(z)$ respectively. Furthermore let M^T denote the transpose of any given matrix M. Then the determinants Δ_k of Th. (43,1) may be written symbolically as

$$\Delta_k = \begin{vmatrix} A_k^T, & \bar{A}_k^* \\ A_k^{*T}, & \bar{A}_k \end{vmatrix} \qquad \text{[Cohn 1].}$$

2. Let A_k^U represent the matrix obtained from A_k by interchanging the jth and $(k - j)$th rows, $j = 1, 2, \cdots, k$. Then, if $f(z)$ is a real polynomial,

$$\Delta_k = | A_k^* + A_k^U | \cdot | A_k^* - A_k^U | \qquad \text{[Cohn 1].}$$

3. The determinant Δ_n is the resultant of the two polynomials $f(z)$ and $f^*(z)$ and hence vanishes if and only if $f(z)$ has a zero on the circle $| z | = 1$. Hint: The resultant of two nth degree polynomials $f(z)$ and $g(z)$ may be written (cf. Bôcher's *Algebra,* New York 1924, pp. 198-9) in terms of the corresponding triangular matrices A_n and B_n as

$$R(f, g) = \begin{vmatrix} \bar{A}_n^*, & A_n^T \\ \bar{B}_n^*, & B_n^T \end{vmatrix} \qquad \text{[Cohn 1].}$$

44. **Polynomials with zeros on the unit circle.** In Th. (42,1) and Th. (43,1) we assumed that $f(z)$ had no zeros on the circle $| z | = 1$ and that none of the δ_j or Δ_j, $j = 1, 2, \cdots, n$, were zero.

Let us consider the effect of removing the first restriction. If $f(z)$ has on the circle $| z | = 1$ only the zeros $e^{\theta_j i}$, $j = 1, 2, \cdots, n - k$, then it has as factor the polynomial

$$(44,1) \qquad \psi(z) = \prod_{j=1}^{n-k} (z - e^{i\theta_j}).$$

Corresponding to this polynomial, we have by eq. (42,4)

$$(44,2) \qquad \psi^*(z) = (-1)^{n-k} e^{-i\sigma} \psi(z)$$

where $\sigma = \theta_1 + \theta_2 + \cdots + \theta_{n-k}$. If the other factor of $f(z)$, the one not zero on $|z| = 1$, is written as

$$g(z) = b_0 + b_1 z + \cdots + b_k z^k, \tag{44,3}$$

then

$$f(z) = \psi(z)g(z) = \prod_{j=1}^{n-k} (z - e^{i\theta_j}) \sum_{j=0}^{k} b_j z^j. \tag{44,4}$$

Since $f^*(z) = \psi^*(z)g^*(z)$,

$$f^*(z) = (-1)^{n-k}e^{-i\sigma}\psi(z)g^*(z). \tag{44,5}$$

That is to say, $\psi(z)$ is a common factor of $f(z)$ and $f^*(z)$. Conversely if $\psi(z)$ is a common factor of $f(z)$ and $g(z)$, all the zeros of $\psi(z)$ must clearly lie on the circle $|z| = 1$.

The polynomial $\psi(z)$ may be found by applying to $f(z)$ and $f^*(z)$ the Euclid algorithm for finding their greatest common divisor. But $\psi(z)$ may also be found by use of the sequence (42,7) of polynomials $f_k(z)$. In fact, let us denote by $g_k(z)$ the sequence (42,7) in which $f(z)$ is replaced by $g(z)$. Then, since in (44,4) $a_0 = (-1)^{n-k}e^{i\sigma}b_0$ and $a_n = b_k$, it follows that

$$f_1(z) = (-1)^{n-k}e^{-i\sigma}\bar{b}_0\psi(z)g(z) - b_k(-1)^{n-k}e^{-i\sigma}\psi(z)g^*(z)$$

and, hence, that

$$f_1(z) = (-1)^{(n-k)}e^{-i\sigma}\psi(z)g_1(z). \tag{44,6}$$

Similarly

$$\begin{aligned} f_2(z) &= (-1)^{2(n-k)}e^{-2i\sigma}\psi(z)g_2(z), \\ &\cdots\cdots\cdots \\ &\cdots\cdots\cdots \\ &\cdots\cdots\cdots \\ f_k(z) &= (-1)^{k(n-k)}e^{-ki\sigma}\psi(z)b_0^{(k)}, \\ f_{k+1}(z) &\equiv 0. \end{aligned} \tag{44,7}$$

In other words, if $f(z)$ and $f^*(z)$ have a common factor $\psi(z)$ of degree $n - k$, it is a factor of all the $f_j(z)$, $j = 1, 2, \cdots, k$, and $f_{k+1}(z) \equiv 0$ together with $\delta_j = 0, j > k$.

Conversely, if

$$f_{k+1}(z) = \bar{a}_0^{(k)}f_k(z) - a_{n-k}^{(k)}f_k^*(z) \equiv 0, \tag{44,8}$$

we may show that $f_k(z)$ is a factor common to all the $f_j(z)$ and $f_j^*(z)$, $j = k - 1, k - 2, \cdots, 1$, and to $f(z)$ and $f^*(z)$. For, from eq. (42,7) we obtain

$$zf_{j+1}^*(z) = a_0^{(j)}f_j^*(z) - \bar{a}_{n-j}^{(j)}f_j(z),$$

and thus

$$(44,9)\qquad \begin{aligned} \delta_{j+1}f_j(z) &= a_0^{(j)}f_{j+1}(z) + a_{n-j}^{(j)}zf_{j+1}^*(z), \\ \delta_{j+1}f_j^*(z) &= \bar{a}_{n-j}^{(j)}f_{j+1}(z) + \bar{a}_0^{(j)}zf_{j+1}^*(z). \end{aligned}$$

If we substitute from (44,8) into (44,9) with $j = k - 1$, we find from the equations

$$\delta_k f_{k-1}(z) = [a_0^{(k-1)} + a_{n-k+1}^{(k-1)}(\bar{a}_0^{(k)}/a_{n-k}^{(k)})z]f_k(z),$$

$$\delta_k f_{k-1}^*(z) = [\bar{a}_{n-k+1}^{(k-1)} + \bar{a}_0^{(k-1)}(\bar{a}_0^{(k)}/a_{n-k}^{(k)})z]f_k(z),$$

that $f_k(z)$ is a common factor of $f_{k-1}(z)$ and $f_{k-1}^*(z)$. By application of eqs. (44,9) with $j = k - 1, k - 2, \cdots, 0$, we may also show $f_k(z)$ is a factor of $f(z)$ and $f^*(z)$.

The number of zeros of $f(z)$ in the circle $|z| < 1$ is the same as the number of zeros of $g(z)$ in this circle. If we set $\epsilon_j = b_0^{(j)}$, the ϵ_j are the δ_j for $g(z)$ and so the number of zeros of $f(z)$ in $|z| < 1$ is the number of negative products $(\epsilon_1\epsilon_2 \cdots \epsilon_j)$, $j = 1, 2, \cdots, k$. Since from eqs. (44,1) and (44,7) we find

$$a_0^{(j)} = (-1)^{(j+1)(n-k)}e^{-(j+1)i\sigma}b_0^{(j)},$$

$$a_{n-j}^{(j)} = (-1)^{j(n-k)}e^{-ji\sigma}b_{k-j}^{(j)},$$

$$\delta_{j+1} = |a_0^{(j)}|^2 - |a_{n-j}^{(j)}|^2 = |b_0^{(j)}|^2 - |b_{k-j}^{(j)}|^2 = \epsilon_{j+1}.$$

In other words, we have proved the following generalization of Th. (42,1) due to Marden [16].

THEOREM (44,1). *For a given polynomial $f(z) = a_0 + a_1z + \cdots + a_nz^n$, let the sequence* (42,7) *of polynomials $f_j(z)$ be constructed. Then, if, for some $k < n$, $P_k \neq 0$ in eq.* (42,16) *but $f_{k+1}(z) \equiv 0$, then $f(z)$ has $n - k$ zeros on the unit circle $|z| = 1$ at the zeros of $f_k(z)$; it has p zeros in this circle, where p is the number of negative P_j for $j = 1, 2, \cdots, k$, and it has $q = k - p$ zeros outside this circle.*

EXERCISES. Prove the following.

1. The number p in Th. (44,1) may be taken as the number of variations of sign in the sequence $1, \Delta_1, \Delta_2, \cdots, \Delta_k$ of determinants (43,1) [Marden 16; cf. Cohn 1, p. 129].

2. Let $f(z)$ be a real polynomial of degree n and let

$$g(z^2) = (z^2 + 1)^n f((z + i)/(z - i))f((z - i)/(z + i)).$$

Then $f(z)$ has k zeros on the circle $|z| = 1$ if and only if $g(z)$ has k positive real zeros [Kempner 2, 3 and 7].

45. Singular determinant sequences. Returning to polynomials $f(z)$ which do not have any zeros on the circle $|z| = 1$, let us consider the case that, for some $k < n$, $\delta_1\delta_2 \cdots \delta_k \neq 0$ but

$$\delta_{k+1} = a_0^{(k+1)} = |a_0^{(k)}|^2 - |a_{n-k}^{(k)}|^2 = 0. \tag{45,1}$$

In such a case the number p of zeros of $f(z)$ in the unit circle $C : |z| < 1$ may be found either by a limiting process or by a modification of the sequence (42,7).

The limiting process may be chosen as one operating upon the circle C or upon the coefficients of $f_k(z)$. That is, since $f_k(z)$ has no zeros on the circle C, we may consider in place of $f_k(z)$ the polynomial

$$F_k(z) = f_k(rz) \tag{45,2}$$

which, for $r = 1 \pm \epsilon$ and ϵ a sufficiently small positive quantity, has as many zeros in the circle $|z| < 1$ as does $f_k(z)$. Alternatively, we may consider in place of $f_k(z)$ the polynomial

$$F_k(z) = (1 + \epsilon)a_0^{(k)} + \sum_{j=1}^{n-k} a_j^{(k)}z^j = \epsilon a_0^{(k)} + f_k(z), \tag{45,3}$$

which for ϵ a sufficiently small real number has also as many zeros in the unit circle as does $f_k(z)$.

A more direct procedure for covering the case of a vanishing δ_{k+1} is to modify the sequence (42,7).

Recognizing that, according to (45,1), in

$$f_k(z) = a_0^{(k)} + a_1^{(k)}z + \cdots + a_{n-k}^{(k)}z^{n-k} \tag{45,4}$$

the first and last coefficients have equal modulus, we shall find useful the following two theorems due to Cohn [1].

THEOREM (45,1). *If the coefficients of the polynomial* $g(z) = b_0 + b_1z + \cdots + b_mz^m$ *satisfy the relations:*

$$b_m = u\bar{b}_0\,, \qquad b_{m-1} = u\bar{b}_1\,, \cdots, b_{m-q+1} = u\bar{b}_{q-1}\,, \qquad b_{m-q} \neq u\bar{b}_q \tag{45,5}$$

where $q \leqq m/2$ *and* $|u| = 1$, *then* $g(z)$ *has in the circle* $|z| < 1$ *as many zeros as the polynomial*

$$G_1(z) = \bar{B}_0G(z) - B_{m+q}G^*(z) = \sum_{j=0}^{m} B_j^{(1)}z^j, \tag{45,6}$$

where

$$G(z) = (z^q + 2b/|b|)g(z) = \sum_{j=0}^{m+q} B_jz^j, \tag{45,7}$$

$$b = (b_{m-q} - u\bar{b}_q)/b_m\,, \tag{45.8}$$

and $|B_0^{(1)}| < |B_m^{(1)}|$.

THEOREM (45,2). *If the coefficients of the polynomial* $g(z) = b_0 + b_1 z + \cdots + b_m z^m$ *satisfy the relations*

$$(45,9) \qquad b_m = u\bar{b}_0\,, \qquad b_{m-1} = u\bar{b}_1\,, \cdots, b_0 = u\bar{b}_m\,,$$

then $g(z)$ *has in the circle* $|z| < 1$ *as many zeros as the polynomial*

$$(45,10) \qquad g_1(z) = [g'(z)]^* = z^{m-1}\bar{g}'(1/z) = \sum_{j=0}^{m-1} b_j^{(1)} z^j$$

where $g'(z)$ *is the derivative of* $g(z)$.

Since polynomial $f_k(z)$ in (45,4) is a polynomial $g(z)$ of the type in either Th. (45,1) or Th. (45,2), these theorems permit the replacement of $f_k(z)$ by a polynomial which is of degree not exceeding $n - k$ and in which relations (45,5) and (45,9) are not satisfied.

Let us first prove Th. (45,1). As the factor $(z^q + 2b/|b|)$ does not vanish in the unit circle $|z| = 1$, $g(z)$ has as many zeros $|z| < 1$ as $G(z)$. Since, however, $B_0 = 2(b/|b|)b_0$ and $B_{m+q} = b_m$, we learn from (45,5) that

$$D_1 = |B_0|^2 - |B_{m+q}|^2 = 2|b_0|^2 - |b_m|^2 = |b_m|^2 > 0,$$

and, from (42,10) with $n - j$ replaced by m and δ_{j+1} replaced by D_1, we learn that $G(z)$ and $G_1(z)$ have the same number of zeros in $|z| = 1$. If we compute $G_1(z)$ by use of eqs. (45,6) and (45,7), we find $B_j^{(1)} = 0$ if $j > m$; that is, $G_1(z)$ is a polynomial of the same degree as $g(z)$. Also,

$$B_0^{(1)} = 4|b_0|^2 - |b_m|^2 = 3|b_0|^2,$$

$$B_m^{(1)} = b_0 b_m(2|b| + 3) = |b_0|^2 u(2|b| + 3)$$

and thus $|B_0^{(1)}| < |B_m^{(1)}|$.

Let us next prove Th. (45,2). We begin by proving that for any sufficiently small positive number ξ, the polynomial

$$(45,11) \qquad G(z) = g(z) - \xi z g'(z)$$

has as many zeros in the circle $|z| = 1$ as does $g(z)$. By Th. (1,4), this follows from the continuity of the zeros of $G(z)$ as functions of ξ, provided $g(z)$ has no zeros on $|z| = 1$. If $g(z)$ has on $|z| = 1$ the zero α of multiplicity μ, then $G(z)$ will have μ zeros in the neighborhood of $z = \alpha$. We wish to show that, for sufficiently small ξ ($\xi > 0$), these μ zeros may be made to lie on or exterior to the circle $|z| = 1$.

For this purpose, let us write

$$g(z) = (z - \alpha)^\mu h(z), \qquad \mu \geqq 1, \qquad h(\alpha) \neq 0.$$

Then, from (45,11)

$$G(z) = (z - \alpha)^{\mu - 1}[H_1(z) - H_2(z)],$$

where

$$H_1(z) = [z - \alpha - \xi z\mu]h(z); \qquad H_2(z) = \xi z(z - \alpha)h'(z).$$

Let us draw a circle $\Gamma(\xi)$ with center $\beta = (1 + \mu\xi)\alpha$ and radius $(|\alpha|/2)\xi$. This circle lies exterior to the unit circle. If we choose ξ sufficiently small ($\xi > 0$), then $h(z) \neq 0$ in and on circle $\Gamma(\xi)$ but the bracket in $H_1(z)$ will vanish at the point

$$z = \frac{\alpha}{1 - \mu\xi} = (1 + \mu\xi)\alpha + \frac{\alpha\mu^2\xi^2}{1 - \mu\xi}$$

which, if $0 < \lambda < 1$ and $0 < \xi < \lambda/(\mu + 2\mu^2)$, will lie inside the circle $\Gamma(\xi)$. Hence, the polynomial $H_1(z)$ has just one zero in the circle $\Gamma(\xi)$. Furthermore, at any point

$$\zeta = (1 + \mu\xi)\alpha + e^{i\theta}(|\alpha|/2)\xi$$

on the circle $\Gamma(\xi)$, we find

$$H_1(\zeta) = \xi[e^{i\theta}(|\alpha|/2)(1 - \xi\mu) - \alpha\xi\mu^2]h(\zeta),$$

$$H_2(\zeta) = \xi^2[(1 + \mu\xi)\alpha + e^{i\theta}(|\alpha|/2)\xi][\mu\alpha + e^{i\theta}|\alpha|/2]h'(\zeta).$$

That is, on the circle $\Gamma(\xi)$, since $h'(\zeta)/h(\zeta)$ is bounded, $H_1(\zeta)$ and $H_2(\zeta)$ are of the orders of magnitude of ξ and ξ^2 respectively. In other words, there is a choice of λ such that $|H_2(\zeta)| < |H_1(\zeta)|$ on all circles $\Gamma(\xi)$ and, hence, by Rouché's Theorem, $[H_1(z) - H_2(z)]$ has the same number of zeros in circle $\Gamma(\xi)$ as $H_1(z)$. This means that for any sufficiently small positive value ξ, $G(z)$ has no additional zeros interior to the circle $|z| = 1$, to correspond to the zero α which $g(z)$ had on the unit circle. That is to say, for any sufficiently small positive number ξ, $G(z)$ has the same number of zeros in the unit circle as does $g(z)$.

Let us now write the polynomial (45,11) as

$$G(z) = \sum_{j=0}^{m} (1 - \xi j)b_j z^j = \sum_{j=0}^{m} B_j z^j,$$

noting in particular that

$$B_0 = b_0 \qquad \text{and} \qquad B_m = (1 - m\xi)b_m\,.$$

By hypothesis $|b_0| = |b_m|$. If $0 < \xi < 1/m$, then

$$D_1 = |B_0|^2 - |B_m|^2 > 0.$$

According to Lemma (42,1), the polynomial

$$G_1(z) = \bar{B}_0 G(z) - B_m G^*(z) \tag{45,12}$$

has as many zeros in the circle $|z| = 1$ as does $G(z)$ and therefore as does $g(z)$. If we substitute from eq. (45,11) into eq. (45,12), we find

$$G_1(z) = \bar{b}_0[g(z) - \xi z g'(z)] - (1 - m\xi)b_m[g^*(z) - \xi(g'(z))^*]$$

$$= \bar{b}_0 g(z) - b_m g^*(z) + \xi[m b_m g^*(z) - z\bar{b}_0 g'(z)] + \xi(1 - m\xi)b_m[g'(z)]^*.$$

By use of the relations (45,9), we find that

$$\bar{b}_0 g(z) \equiv b_m g^*(z),$$

$$\bar{b}_0 g'(z) = b_m[g^*(z)]' = b_m[z^m \bar{g}(1/z)]'$$

$$= (b_m/z)\{mg^*(z) - [g'(z)]^*\},$$

and thus that

$$b_m[g'(z)]^* = mb_m g^*(z) - z\bar{b}_0 g'(z).$$

These relations permit us to reduce $G_1(z)$ to

$$G_1(z) = \xi[2 - m\xi]b_m[g'(z)]^*.$$

That is to say, as required for the proof of Th. (45,2), we have shown that $[g'(z)]^*$ has the same number of zeros in the unit circle as does $g(z)$.

In summary, we may say that, if the first and last coefficients of the $f_k(z)$ in (45,4) have the same modulus, then by applying Th. (45,1) or Th. (45,2) we may replace $f_k(z)$ by another polynomial having the same number of zeros in the circle $|z| = 1$ as does $f_k(z)$, but having first and last coefficients of unequal modulus. This replacement permits us to resume the computation of the δ_j inasmuch as the new δ_{k+1} is not zero.

Exercises. Prove the following.

1. Let $\Delta_k(r)$ be the value of the determinant Δ_k of Th. (43,1) for $a_k = b_k r^k$, $k = 0, 1, \cdots, n$. Then the polynomial $g(z) = b_0 + b_1 z + \cdots + b_n z^n$ has in the circle $|z| = r$ the number $p(r)$ of zeros and in the ring $r_1 < |z| < r_2$ the number $m(r_1, r_2)$ of zeros, these numbers being

$$p(r) = \mathfrak{V}\{1, \Delta_1(r), \Delta_2(r), \cdots, \Delta_n(r)\},$$

$$m(r_1, r_2) = \mathfrak{V}\{1, \Delta_1(r_2), \Delta_2(r_2), \cdots, \Delta_n(r_2)\}$$

$$- \mathfrak{V}\{1, \Delta_1(r_1), \Delta_2(r_1), \cdots, \Delta_n(r_1)\}.$$

It is assumed that all the $\Delta_k(r)$, $\Delta_k(r_1)$ and $\Delta_k(r_2)$ are different from zero [Cohn 1].

2. If in the sequence (42,7) $f_k(z)$ is the first $f_j(z)$ of the type $g(z)$ in Th. (45,2), and if the polynomial $f_{k+1}(z) = z^{n-k-1} f_k'(1/z)$ has m zeros in the unit circle, then $f_k(z)$ and $f(z)$ have each $[n - k - 2m]$ zeros on the unit circle [Cohn 1].

3. A necessary and sufficient condition for all the zeros of $g(z)$ to lie on the unit circle is that $g(z)$ satisfy conditions (45,9) and that all the zeros of $g'(z)$ lie in this circle [Cohn 1].

4. A necessary and sufficient condition that all the zeros of $f(z) = a_0 + a_1 z + \cdots + a_n z^n$ lie on the circle $|z| = 1$ is that in eq. (42,8) all $a_k^{(1)} = 0$ and that also $f'(z)$ have all its zeros in this circle [Schur 2].

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